Effectiveness of $\mathcal{A}$-Hypergeometric Series for Approximating Positive Real Roots of Trinomials

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Abstract

Given a univariate trinomial of arbitrary degree, $\mathcal{A}$-Hypergeometric can be used to find its approximate roots. Three families of series are explicitly known, however in certain cases these series must use large numbers of terms to obtain an undesirably low number of digits of accuracy. A new pair series has been constructed to account for these cases. This report uses Maple 2023 to compare the effectiveness of these series at approximating roots to a certain number of digits in varying instances.

1 Introduction

Quickly finding the roots of polynomials is of great interest and importance in many applications. For univariate polynomials of arbitrary degree, this problem is trivial for the binomial case but becomes more complex for trinomials. Up to degree 4, explicit radical formulas are known which can find these roots based on coefficients, but such formulas do not exist for degree 5 or higher.

Three families of Puiseux series, discussed in Section 2, as a function of the middle coefficient of a given trinomial are explicitly known that converge near the roots of trinomials of arbitrary degree. The approximations of these roots obtained by these series can then be applied in Newton iterations to obtain more precise estimations of the roots. However, these series require many more terms to obtain a reasonable number of accurate decimal places when this coefficient is within a certain range, leading to the development of a new pair of series, for which an explicit form is not currently known.

This report uses the term "effective" to describe an instance where a given series can approximate a root to a specified number decimal places with a reasonably few number of terms. This report compares the effectiveness of two of these series at varying values of the middle coefficient.

2 Background

Trinomials and Deterministic Complexity Limits for Real Solving [1] details how for a trinomial of arbitrary degree, its roots can be expressed as an explicit Puiseux series. A summary of these findings is explained in this section. Considering an equation of the form:

$$f_c(x) = 1 - cx^m + x^n$$

where $m, n \in \mathbb{N}$, $0 < m < n$, $gcd(m, n) = 1$, and $c \neq 0$ there are three cases and families of series corresponding to each case that give the positive roots of f_c. We define the singularity as $r_{m,n} := \left|\frac{n}{m^{m/n}(n-m)^{(n-m)/n}}\right|$. The series used to find the positive roots of f_c is dependent on the value of middle coefficient c when compared to $r_{m,n}$.

2.1 Case 1: $|c| < r_{m,n}$

When $|c| < r_{m,n}$, the following series $x_{mid}(c)$ gives the positive root of f_c:

$$x_{mid}(c) = (-1)^{1/n}\left[1 + \sum_{k=1}^{\infty}\left(\frac{1}{km^k} \cdot \prod_{j=1}^{k-1} \frac{1 + km - jn}{j}\right)((-1)^{(m+n)/n}c)^k\right]$$

2.2 Case 2: $|c| > r_{m,n}$

When $|c| > r_{m,n}$, the function f_c has two positive roots. The smaller root is given by the following series $x_{lo}(c)$:

$$x_{low}(c) = \frac{1}{c^{1/m}}\left[1 + \sum_{k=1}^{\infty}\left(\frac{1}{km^k} \cdot \prod_{j=1}^{k-1} \frac{1 + kn - jm}{j}\right)\frac{1}{c^{kn/m}}\right]$$

The larger root of f_c is given by the following series $x_{hi}(c)$:

$$x_{hi}(c) = c^{1/(n-m)}\left[1 - \sum_{k=1}^{\infty}\left(\frac{1}{k(n-m)^k} \cdot \prod_{j=1}^{k-1} \frac{km + j(n-m) - 1}{j}\right)\frac{1}{c^{kn/(n-m)}}\right]$$

2.3 Case 3: $|c| \approx r_{m,n}$

When $|c|$ approaches the singularity, the previously stated families of series become much less effective. As such a new series has been found that absolves this issue. The pair of series is given by:

$$x^{\pm}_{sing}(c) = \zeta_{m,n} \sum_{k=0}^{\infty} \frac{\gamma_k}{(\pm\sqrt{(n-m)r})^k}(c - r_{m,n})^{k/2}$$

where $\zeta_{m,n} = \left(\frac{m}{n-m}\right)^{1/n}$ is the unique degenerate positive root of f_c for $c > 0$ and γ_k are rational numbers in $\mathbb{Q}[m, n]$.

The γ_k coefficients are obtainable numerically, and several have been derived symbolically in terms of m and n. For instance, by examining the power series expansion of $f_{w^2+r_{m,n}}(\zeta_{m,n}+\alpha_1^{\pm}w+\alpha_2^{\pm}w^2+...)$ by letting $\alpha_k^{\pm}=\left(\pm\sqrt{\frac{2}{m(n-m)r}}\right)^k\zeta\gamma_k$ and solving for the coefficients of w^2 and w^3 one can find that $\gamma_0=\gamma_1=1$ and $\gamma_2=\frac{2m-n+3}{3}$. This approach is slow and does not provide an explicit formula for all coefficients akin to the existing ones for the coefficients for $x_{mid}(c)$, $x_{hi}(c)$, and $x_{low}(c)$.

3 Results

Example. Given the trinomial

$$f(x)=1-cx^2+x^{13}$$

when $c=1.5362173$ which is close to the singularity $r_{2,13}\approx 1.5362171$, the series $x_{hi}(c)$ and $x_{lo}(c)$ which we should expect to use in this case take more than 20,000 terms to give 3 decimal places of accuracy for a root of f. However, the new pair of series $x_{sing}^{\pm}(c)$ yields 3 digits of accuracy with just 1 term, 7 digits with 2 terms, and so on, despite a complete explicit form not being known yet.

This section details experiments done via Maple 2023 to determine how many terms from two series are needed to obtain a certain level of accuracy for approximating the roots of f. The following tables list the minimum number of terms needed from the series x_{hi} and x_{sing}^{+} to get 3 decimal places of accuracy for the larger positive root of $f(x)=1-cx^m+x^{13}$ for $m=2...6$ at different values of c with increasing distance from the singularity point $r_{m,13}$.

Although higher bounds can be found for certain cases using further experimentation, if more than 10,000 terms are required to get 3 decimal places of accuracy for a particular instance, the precise bound is not given in this report and is simply listed as >10,000 terms.

The below table shows the results of the experiment for the $m=2$ case. Here, rounded to 7 decimal places, $r_{m,n}=1.5462171$.

$f(x) = 1 - cx^2 + x^{13}$		
c	x_{hi}	x_{sing}^+
1.5362173	>10000	2
1.536218	>10000	2
1.53622	>10000	2
1.5363	3202	2
1.537	845	2
1.54	283	2
1.6	49	3
1.7	17	5
1.8	14	5
1.9	22	6
2.0	8	5
2.1	8	6
2.2	8	6
2.3	8	6
2.4	6	9
2.5	5	9

The below table shows the results of the experiment for the $m = 3$ case. Here, rounded to 7 decimal places, $r_{m,n} = 1.7163572$.

$f(x) = 1 - cx^3 + x^{13}$		
c	x_{hi}	x_{sing}^+
1.7163574	>10000	2
1.716358	>10000	2
1.71636	>10000	2
1.7164	>10000	2
1.717	900	2
1.72	319	2
1.8	29	3
1.9	24	4
2.0	9	5
2.1	7	5
2.2	6	5
2.3	9	6
2.4	6	6
2.5	5	6
2.6	7	7
2.7	6	7

The below table shows the results of the experiment for the $m = 4$ case. Here, rounded to 7 decimal places, $r_{m,n} = 1.8538077$.

$f(x) = 1 - cx^4 + x^{13}$		
c	x_{hi}	x^+_{sing}
1.8538079	>10000	2
1.853808	>10000	2
1.85381	>10000	2
1.8539	2409	2
1.854	4687	2
1.86	194	2
1.9	114	4
2.0	15	3
2.1	11	3
2.2	12	4
2.3	7	5
2.4	7	5
2.5	7	5
2.6	7	5
2.7	5	5
2.8	5	5

The below table shows the results of the experiment for the $m = 5$ case. Here, rounded to 7 decimal places, $r_{m,n} = 1.9469780$.

$f(x) = 1 - cx^5 + x^{13}$		
c	x_{hi}	x^+_{sing}
1.9469782	>10000	2
1.946979	>10000	2
1.94698	>10000	2
1.947	4716	2
1.948	747	2
1.95	1256	2
2.0	30	2
2.1	17	2
2.2	10	4
2.3	7	6
2.4	8	5
2.5	11	5
2.6	5	5
2.7	6	4
2.8	5	5
2.9	3	6

The below table shows the results of the experiment for the $m = 6$ case. Here, rounded to 7 decimal places, $r_{m,n} = 1.9940857$.

$f(x) = 1 - cx^6 + x^{13}$		
c	x_{hi}	x^+_{sing}
1.9940859	>10000	2
1.994086	>10000	2
1.99409	>10000	2
1.9941	>10000	2
1.995	1382	2
2.0	138	3
2.1	20	3
2.2	12	3
2.3	7	4
2.4	6	4
2.5	6	4
2.6	8	4
2.7	4	4
2.8	4	4
2.9	4	4
3.0	3	6

4 Discussion

There are general trends in the data obtained that align with what we would expect based on what is already known about how these series tend to work. Across all cases, x_{hi} tends to be very ineffective close to the singularity point $r_{m,n}$, often requiring more than 10,000 terms to achieve the stated level of accuracy, but gradually gets more effective once c moves farther away from that point. Conversely, x^+_{sing} works very well when c is close to $r_{m,n}$ but slowly becomes less effective as c grows larger away from the singularity.

There are many instances where the series do not behave this way. For example, in the $m = 2$ case, when $c = 1.8$ the series x_{hi} requires a minimum of 14 terms to obtain 3 decimal places of accuracy for the root, but $c = 1.9$ now requires 22 terms. Exceptions to the general trend also occur with x^+_{sing}, such as when in the $m = 2$ case c increases from 1.9 to 2.0, the series goes from needing a minimum of 6 terms to only needing 5 terms for this level of accuracy. Similar observations can be seen in cases for other values of m.

There are a few extreme instances where the number of terms needed for the selected level of accuracy changes very unpredictably. In particular, in the $m = 4$ case when $c = 1.85381$ the series x_{hi} needs a minimum of at least 10,000 terms to approximate the larger root of f to 3 decimal places. When increasing c to 1.8539 this series now only needs 2409 terms, but increasing again to 1.854 it now needs a minimum of 4687 terms. A similar instance occurs in the $m = 5$ case. This issue may be a result of only requiring 3 decimal places of accuracy, but further experimentation would be needed to verify this.

Future research on this subject might include similar experiments with more precise increments of m, experiments requiring more decimal places of accuracy, or experiments with trinomials of different values of m and n. Once obtaining more data, future research might also produce precise intervals on which individual series are most effective for a given trinomial.

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References

[1] Erick Boniface, Weixun Deng, and J. Maurice Rojas. 2022. Trinomials and Deterministic Complexity Limits for Real Solving. In ISSAC 2022: International Symposium on Symbolic & Algebraic Computation. ACM, New York, NY, USA, 8 pages