Buzz Contest Rules and Keywords

1 Introduction

Contestants take turns in rotation. The group of contestants is *counting* out loud, starting with 1, each person saying the next number when it comes his turn. Anyone who says the wrong number is eliminated, and the survivors continue. When someone makes a mistake, the next player says the number that is actually next in line, the one the loser should have said. If two contestants are eliminated in a row, the contest emcee will remind the players what number we're at. This gives an unfair advantage to the third contestant to try at that number—but who gets this unfair advantage is pretty random, so it's fair after all, in a way.

This goes on until only the winner remains.

Well, not quite. That would be boring. Sometimes, instead of saying the number that is due to be named, you instead say each of the *buzzwords* that apply to the number. If a contestant says a word that does not apply, or leaves out a word that does, he is out. There is one little endgame wrinkle: when only two contestants remain, if one makes a mistake, he has not yet lost. The other player has to say the correct list of buzzwords, or the correct number if no buzzwords apply. They go back and forth on that number until one gets it right and wins.

The buzzwords are *buzz, bang, crash, whiz, zip, pop, fibbi*, and *squawk.* Not all words are in force from the beginning; the game emcee will say when a word comes into force.

The heart of the matter is the rules for when a buzzword applies to a number. The first few buzzwords are simple enough, but then it gets more complicated.

2 Buzzwords

- *Buzz.* The number is a multiple of 7, or one of its digits is a 7.
- *Bang.* The number is a multiple of 5, or one of its digits is a 5.
- *Crash.* The number is *prime.* The primes are the numbers

  $$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, \cdots$$

  that cannot be written as the product of two whole numbers, both greater than 1.
- *Whiz.* The number is *squarefree.* That is, it may not be prime, but it's not divisible by any square other than 1. Thus, the squarefree numbers are

  $$1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, 31, 33, 34, 35, 37, \cdots.$$

  Never say "crash" without also saying "whiz", because all primes are squarefree.
- *Zip.* The number is a *power.* That is, the number is got by multiplying some other number times itself more than once. Thus, the powers are

  $$1, 4, 8, 9, 16, 25, 27, 32, 36, 49, \cdots.$$

- *Pop.* The number is the *product of two distinct primes.* The pop numbers are

  $$6, 10, 14, 15, 21, 22, 26, 33, 34, 35, \cdots$$

- *Fibbi.* The number is part of the *Fibonacci* sequence. This sequence begins with 0,1 and self-extends by the rule that the next number is the sum of the two that came just before it. So the Fibonacci sequence starts off with

  $$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 \cdots$$

  There are only a few Fibonacci numbers small enough to be likely to occur in play, so memorizing the beginning of the sequence is practical.
- *Squawk.* A squawk number is one that can be written as the sum of exactly two squares. A square is a number that is got by multiplying a whole number times itself. So 0 is a square, so the squares themselves are squawk numbers. ($9 = 9 + 0$, for instance.) The squawk numbers can be a bit difficult to get a handle on. There are a lot of them, so memorizing the list would be a real pain. (It starts out

  $$1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, 25, 26, 29, 32, 34, 36, 37 \cdots$$

  When thinking about whether, say, 343 is the sum of squares, one has to check all too many possibilities: 343? $342 + 1$? $339 + 4$? $334 + 9$? And on and on, and the clock runs out and you lose before being able to check them all. Isn't there any better way? Well, yes.

3 Better Ways

For *powers* [zip], it helps to think along the lines of, powers of what? Positive powers of 2 go $2, 4, 8, 16, cdots$ and you probably know several more in the list. The last one that is at all likely to occur in a buzz contest is probably 256. Powers of 3 go $3, 9, 27, 81, 243$ and out. Powers of 5, $5, 25, 125$ and out, then $7, 49, 343$. After that, only squares are a threat, and it's not that hard to keep in mind the squares.

$$1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169$$

Well, maybe it is hard to keep them all in mind. Is there a way to extend the sequence without having to work out now 14×14 in your head? Yes: Going from 11^2 to 12^2 added 23 to 121, and $23 = 2 * 11 + 1$. Next time, you add $2 * 12 + 1$, and after that, you add $2 * 13 + 1$, and so the next square after 169 is $169 + 27 = 189 + 7 = 196$, and the next after that is $196 + 29 = 225$, and so on. Why does this work? It's good old algebra:

$$(n+1)^2 = n^2 + 2n + 1$$
$$(n+1)^2 - n^2 = 2n + 1.$$

For *squarefree*[whiz], you have to test for divisibility by a square. But here's a shortcut right away: there's no need to test if a number is divisible by say 36, because if it's divisible by 36 it's divisible by 4 and by 9 and you already checked for both of those. So, you only check for divisibility by the square of a prime! Since 100 is a multiple of 4, testing for divisibility by 4 is just a matter of testing the first two digits. $4, 8, 12, 16, 20, 24, 28$, etc. But there's even a shortcut here. If the tens digit is even or isn't there at all (so it's zero, which is even!), then the ones digit needs to be 0, 4, or 8. If the tens digit is odd, then the ones digit needs to be 2 or 6. Do you see how this works? It's a matter of arithmetic mod 4.

$$10 \equiv 2 \pmod 4$$
$$(2k+1)(10) \equiv 20k + 10 \pmod 4 \equiv 0k + 2 \pmod 4 \equiv 2 \pmod 4.$$

It goes the same way for an even tens digit.

Testing for primality doesn't actually *have* to be a matter of checking possible divisors. There are some amazingly clever ways to figure out whether a number is prime while doing far less arithmetic than would be required to try all the divisors that might go into the number. Unfortunately—those clever ways only begin to pay off when dealing with numbers of 4 digits or more, and the buzz cntest lives in the world of triple digits or less. So, we're stuck with trial division. Here, there are still some shortcuts.

To check if a number is a multiple of 3, you can *add its digits* and check if *that* is a multiple of 3. Thus, 476 is not a multiple of 3, because $4 + 7 + 6 = 17$ and $1 + 7 = 8$ and 8 is not a multiple of 3.

Why does it work? Another *modular arithmetic* thing. See, $10 \equiv 1 \pmod 3$. From the point of view of mod 3, the tens digit, and so the 100's digit and all the rest, are just being added up, without multiplying by 10.

There's a trick for 5, but it's beneath your dignity for me to say what it is.

There's a trick for 11. It's kind of like the trick for 3, and it runs off the fact that $10 \equiv -1 \pmod{11}$. See if you can figure out a rule, along the lines of what we did with 3, to tell whether a number is a multiple of 11. If you're too busy, the answer is at the tail end of this article.

If you figure out the rule, and are sitting there bored, here's a puzzle based on that rule: Come up with a number that is not a multiple of 11, and cannot be changed into a multiple of 11 by changing just one of its digits. Thus, 145 won't do, because changing the 5 to a 3 gives a multiple of 11. Nor will 873, because changing the the 7 to a zero gives a multiple of 11.

Testing for larger primes is kind of a pain, but the good news is that numbers that have only large prime factors are kind of rare. For instance, the first number you might have expected to be prime if you only checked 2,3,5, 7, and 11, is 169, only you know that's a square, so that takes it out to $221 = 13 * 17$.

What about squawk? There's a *really good* trick to this one. The key is to think first about numbers that are *prime* and can be written as the sum of two squares.

Primes are key because if two numbers can be written as a sum of two squares, then so can their product. The reason for this is there's a fairly simple algebra trick: if a pair (a, b) of integers has $a^2+b^2 = m$, and (c, d) has $c^2+d^2 = n$, then some sort of trickery with a, b, c, d gives mn. The simplest instance of this trickery gets you from m to $2m$, and it goes like this: if $a^2 + b^2 = m$, then $(a - b)^2 + (a + b)^2 = 2m$. Mutliply it out, and you get

$$a^2 - 2ab + b^2 + a^2 + 2ab + b^2 = 2a^2 + 2b^2.$$

The general rule is that your new pair, instead of being $(a - b, a + b)$, will be $(ca - db, da + cb)$. So it all comes down to primes.

Here goes with making a list.

$2 = 1 + 1, 5 = 4 + 1, 13 = 9 + 4, 29 = 25 + 4, 37 = 36 + 1, 41 = 25 + 16,$

$53 = 49 + 4, 61 = 36 + 25, 73 = 64 + 9, 89 = 64 + 25, 97 = 81 + 16$

but

$3, 7, 11, 19, 23, 31, 43, 47, 59, 67, 71, 79, 83$ don't work.

Notice anything?

The primes that work, apart from 2 which is a special case, are the numbers congruent to 1 mod 4. The reasons for this are a bit involved, but the reason that the numbers that are 3 mod 4 don't work is really simple from the point of view of mod 4. There are only two types of squares mod 4: those that are 0 mod 4, and those that are 1 mod 4. Add two numbers taken from this meager pool, and you get 0, 1, or 2. You cannot get 3. And that's that.

So, let's say we're wondering about 343. It isn't prime, but here's the thing: numbers that *can* be written as the sum of two squares, when multiplied, give another such number! That's because if $n = a^2 + b^2$, and $m = c^2 + d^2$, then using i for $\sqrt{-1}$, that's like saying $n = (a+ib)(a-ib)$, and $m = (c+id)(c-id)$. But then

$$\begin{aligned} nm &= (a+ib)(a-ib)(c+id)(c-id) = (a+ib)(c+id)(a-ib)(c-id) \\ &= ((ac-bd)+i(ad+bc))((ac-bd)-i(ad+bc)) \\ &= ((ac-bd)^2+(ad+bc)^2. \end{aligned}$$

Huh!? Wait a minute.

Good, let's try an example. We had $13 = 9 + 4$, and $29 = 25 + 4$. So $a = 3$, $b = 2$, $c = 5$, and $d = 2$, and the rule says that $377 = 13*29$ should be writable as the sum of two squares, $(ac-bd)^2 = (15-4)^2 = 121$, and $(ad+bc)^2 = 16^2 = 256$. Ta-dah, $121 + 256 = 377$.

General rule: whenever anything makes your head spin, see if you can trace out what is going on by referring back to an example. Not too terribly simple an example, or it won't bear any clear connection to the wider picture, but not so big and messy that you get lost in the details of the example.

OK, now back to 343. We try dividing by 3 and 5 and it's not a multiple of either of those, and we could try dividing by 2 but that's not going to happen. So 7 into 343? But 343 is 7 less than 350, so 343 is 7*7*7. And 7 is on our bad list of primes. So maybe 343 is NOT a squawk number.

But wait. 49 is a squawk number. Can you come up with a rule to tell WHEN using a bad prime, one that is 3 mod 4, as one of the factors of n, makes it not a sum of squares?

Right. The bad prime has to occur to an odd power. And now, you're fully armed for squawking duels. You don't have to try out all those ways to add this and that square to get the target n. You factor n in your head, going until you hit a prime p that's 3 mod 4. If it divides the number once, divide it into what's left and so on until you know whether p divides n to an even power or an odd power. If it's an odd power, don't say squawk. If you get the number factored and there's no reason like that *not* to say squawk, well, SQUAWK.

Closing puzzle. [Take home, and email me at dhensley@math.tamu.edu if you hit upon something.] Suppose we had a word, say, *scream?*, and it applied to n whenever n can be written in the form $n = a^2 - ab + b^2$. Can you discover a rule, mod something or other, that distinguishes primes p that scream, from those that don't? If two screaming numbers are multiplied, does the product scream? If so, how do you find the a and b for the product, from the a's and b's of the numbers that were multiplied?

And now, the trick for 11: Add digits an odd number of places from the start (either end will do) and subtract the others. Check whether that total is a multiple of 11. For instance, faced with 12345608, you go $1-2+3-4+5-6+0-8 = -11$, and that's a multiple of 11, so 12345608 is a multiple of 11. Or closer to home, 275 is a multiple of 11 because $2-7+5 = 0$ and 0 is a multiple of everything. Including 11. And finally, for that number that isn't a multiple of 11, and cannot be changed to one by altering one digit? Here's one example: 454545454545.

Now, as to why primes p that are 1 mod 4 do allow for writing as $p = a^2 + b^2$. The reason involves two ideas. First, you find an integer n so that $n^2 + 1$ is a *multiple* of p. Well, you construct it. Take $m = (p-1)/2$, and note that m is even because $p-1$ was a multiple of 4. Now take $n = m!$. Thus if $p = 13$, $m = 6$ and $n = 720$. Think about $1*2*3\cdots(p-2)*(p-1)$, mod p. This is congruent to $1*2\cdots*(-2)(-1)$, which simplifies to $(-1)^{(p-1)/2}*n^2 \equiv n^2$. On the other hand, every number x mod p has another number y mod p so that $xy \equiv 1 \pmod{p}$. Thus, $2*7 \equiv 1 \pmod{13}$, and $3*9 \equiv 1 \pmod{13}$, and so on. If we pair things up this way then 1, and -1, pair with themselves, and the others pair off. So $1*2*\cdots*12 \equiv 1*$ paired off stuff that amounts to a bunch of 1's $*(-1)$, and that's -1.

The other main idea is that you've got two vectors: $(n, 1)$ and $(p, 0)$, both with the property that the sum of their squares is a multiple of p. Any combination $s(n, 1) + t(p, 0)$ also works, and you just hunt around for the shortest vector of this sort, and behold, it has length just $\sqrt{p}$.

But this isn't the place for full details.