Solutions Final Exam Math 152, Spring 2007

There are fifteen multiple choice problems and five work-out problems. The multiple choice problems are worth four points each, and the workout problems are worth eight points each, for a total of 100 possible.

1. Find

$$\int_{x=0}^{1} \left(\frac{2x+2}{x^2+2x+3} \right) dx.$$

   (a) ∞

   (b) $-\infty$

   (c) $5/36$

   (d) $\ln(2)$

   (e) $\ln(9) - \ln(4)$

   Answer (d). Substitute $u = x^2 + 2x + 3$, $du = (2x+2)dx$ and figure out the limits on u corresponding to $x = 0$ ($u = 3$), and $x = 1$ ($u = 6$). The integral turns into $\int_3^6 du/u = \ln(6) - \ln(3) = \ln(6/3) = \ln(2)$.

2. The series $\sum_{n=1}^{\infty}(\sqrt{n} - \sqrt{n+1})$

   (a) Diverges because the terms go to infinity.

   (b) Is a telescoping series that converges to 1

   (c) Converges by comparison to a geometric series

   (d) Diverges because the nth partial sum is $1 - \sqrt{n+1}$ which goes to $-\infty$.

   (e) Diverges because both the positive and negative sums are infinite.

   The answer is (d). The second partial sum, for instance, is $(1 - \sqrt{2}) + (\sqrt{2} - \sqrt{3}) = 1 - \sqrt{3}$. As to other answers, (a) is wrong because the terms go to zero, and (e) is wrong because all the terms are negative so there aren't any positive terms to have an infinite sum of.

3.

Find the area between the arc of the curve $x^2 = y^3$ going from $(0,0)$ to $(1,1)$, and the line $y = x$ going from $(0,0)$ to $(1,1)$. as shown:

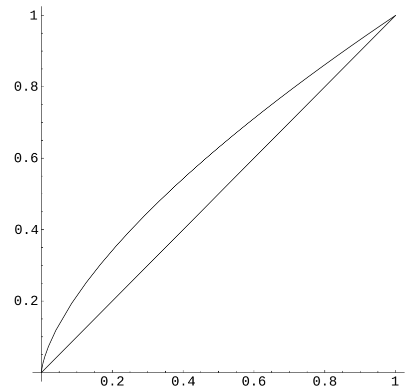
(a) 1/10

(b) 1/9

(c) 1/8

(d) 1/7

(e) 1/6



The answer is (a). Solve for y in $y^3 = x^2$ and you get $y = x^{2/3}$. So the area is given by

$$\int_0^1 x^{2/3} - x\,dx = \frac{3}{5}x^{5/3} - \frac{1}{2}x^2\bigg|_0^1 = \frac{6}{10} - \frac{5}{10} = \frac{1}{10}.$$

4.

Find the volume of the region got by rotating about the x axis the region bounded between the curve $x^2 = y^3$ and the line $y = x$. as shown (two views):

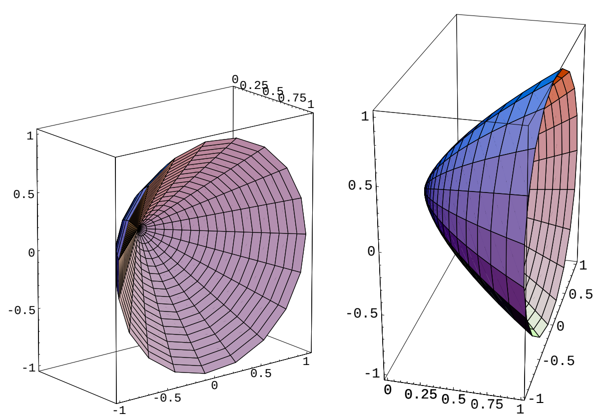
(a) $\pi/25$

(b) $2\pi/21$

(c) $3\pi/17$

(d) $4\pi/13$

(e) $5\pi/9$



The answer is (b). Integrate the area of the washer at cross section x which is $\pi(x^{4/3}) - \pi x^2$. Limits are from 0 to 1, upshot is $2\pi/21$.

5. Find

$$\sum_{n=0}^{\infty} \frac{(-1)^n(\ln(2))^n}{n!}$$

(a) $\ln(\pi)$

(b) $e^{-\pi}$

(c) $\ln(2)$

(d) $\ln(6)$

(e) $1/2$.

This is the series for e^x at $x = -\ln(2)$, so the result is $e^{-\ln(2)} = 1/2$. Answer is (e).

6. The series $\sum_{n=1}^{\infty}(-1)^{n-1}\left(1 - e^{-n}\right)^n$

   (a) Is a geometric series that converges to e^{-n}.

   (b) Converges by the ratio test, though it is not geometric.

   (c) Diverges by the ratio test.

   (d) Converges although the ratio test does not give a verdict one way or the other.

   (e) Diverges although the ratio test does not give a verdict one way or the other.

   This diverges. The terms go to ± 1 because e^{-n} is very small so that essentially one has 1^n. Some care would be needed here if the question had not been multiple choice. A more complete solution would be to observe that $(1-a)^n > 1 - an$ so that $(1-e^{-n})^n > 1 - ne^{-n}$. Now the limit as $n \to \infty$ of $-ne^{-n}$ is zero, so the limit of $(1 - ne^{-n})$ is 1, and our expression is sandwiched between two sequences, one going to 1 and the other sitting at 1 all along. It limits to 1 as well. The ratio test cannot give a verdict of divergent unless the terms go to infinity in absolute value. (And it cannot give a verdict of convergent, unless the terms go to zero exponentially fast.) Here, they don't go to infinity, (nor to zero, for that matter), so the ratio test is not giving a verdict. The answer is (e).

7. Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{(n+1)! + (n+2)!}.$$

(a) 0
(b) 2
(c) 4
(d) 8
(e) ∞

The answer is (e), an infinite radius of convergence. The denominator is larger than with the series for e^x, where it is $n!$ and already large enough to force an infinite radius of convergence. The ratio test also works. The ratio of absolute values of $n+1$st term over nth term is $|x|(n+3)/((n+2)(n+4))$ which goes to zero however large $|x|$ is.

8. Find

$$\int_0^1 \left(\frac{2}{(x+2)(x+4)} \right) dx.$$

(a) $\ln(3) + \ln(5) - \ln(2)$
(b) $-\ln(3) + \ln(2) + \ln(5)$
(c) $(\ln(3) + \ln(2))/\ln(5)$
(d) $\ln(3) + \ln(2) - \ln(5)$
(e) $\ln(3)\ln(2) - \ln(5)$.

The answer is (d). Partial fractions puts the integral into the form

$$\begin{aligned}
&\int_0^1 \left(\frac{1}{x+2} - \frac{1}{x+4} \right) dx \\
&= (\ln(x+2) - \ln(x+4)) \,|_0^1 = \ln(3) - \ln(2) - \ln(5) + \ln(4) \\
&= \ln(3) + \ln(2) - \ln(5).
\end{aligned}$$

9. The arc tangent function, $\tan^{-1}(x)$, is the integral of the series for $1/(1+x^2)$. As a result, the Maclaurin series for $\tan^{-1}(x)$ is

(a) $\sum_{n=0}^{\infty} x^{2n+1}/(2n+1)!$
(b) $\sum_{n=0}^{\infty} (-1)^n x^{2n+1}/(2n+1)!$
(c) $\sum_{n=0}^{\infty} (x^2-1)^n/(2n+1)$
(d) $\sum_{n=0}^{\infty} (-1)^n x^{2n+1}/(2n+1)$
(e) $\sum_{n=0}^{\infty} (x-(2n+1))^n/(2n+1)!$

The answer is (d). You just integrate term by term. The integral of the general term $(-1)^n x^{2n}$ is $(-1)^n x^{2n+1}/(2n+1)$ and you specify adding them up by slapping a summation sign in front, indexed from 0 to infinity.

10. Find $\int_{x=1}^{2} x\ln(x)\,dx$.

    (a) $3\ln(2)$
    (b) $3(2\ln(2)-1)$
    (c) $2\ln(2)-3/4$
    (d) $2\ln(2)$
    (e) $(\ln(2))^2$

    The answer is (c) by a routine integration by parts, beginning with $U = \ln(x)$ and $dV = 1\,dx$, then $dU = 1/x, dx$ and $V = x$.

    For the next several problems, let $\mathbf{a} = \langle 1, 2, 2\rangle$ and $\mathbf{b} = \langle 2, 1, 2\rangle$.

11. Find $\mathbf{a}\cdot\mathbf{b}$.

    (a) 2
    (b) 4
    (c) 6
    (d) 8
    (e) 10

    It's (d), 8.

12. Find $\mathbf{a}\times\mathbf{b}$.

    (a) $\langle 2, 2, -3\rangle$
    (b) $\langle 2, -2, -3\rangle$
    (c) $\langle 2, 2, 4\rangle$
    (d) -12
    (e) 16

    It's (a).

13. Find the distance from $(1, 2, 2)$ to $(2, 1, 2)$.

    (a) $\langle 1, -1, 0\rangle$
    (b) 0
    (c) $\sqrt{2}$
    (d) 2
    (e) $\sqrt{6}$

    It's (c).

14. Find a vector $\mathbf{p}$ that is parallel to $\mathbf{a}$ so that $\mathbf{p} \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{a}$.

(a) $(8/3)\mathbf{a}$

(b) $(3/8)\mathbf{a}$

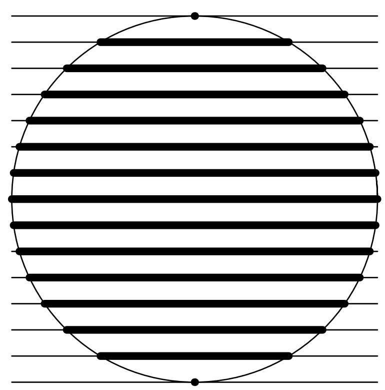
(c) $(8/9)\mathbf{a}$

(d) $(9/8)\mathbf{a}$

(e) $\mathbf{p}/\mathbf{b}$

If you check them one after the other, the first one to work is (c). Answer (e) doesn't even make sense. You cannot divide one vector by another. But there is another way than checking things one by one. You want a vector of the form $c\mathbf{a}$, because all vectors parallel to $\mathbf{a}$ have that form. So now the requirement that $\mathbf{p} \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{a}$ translates to this: $c\mathbf{a} \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{a}$. But we *know* the values of these dot products. They are 9 and 8, respectively. So now it has boiled down to just this: find the right value of c so that $c \cdot 9 = 8$. So $c = 8/9$.

15. Find the average value of the length of a horizontal section of a disk of radius 1. (The thick part of the lines in the diagram, but averaged over not just the ones shown, but all the ones in between.)



(a) $\pi/2$

(b) $\pi \ln(2)$

(c) $\ln(2\pi)$

(d) $2/\pi$

(e) $2\ln(\pi)$

The answer is (a). You integrate the width of the cross sections, and divide by the length of the interval over which you did the integration. But that integration gives the area of a unit circle, so the integral gives π, and the interval of integration was from $y = -1$ to $y = 1$ so it has length 2. That gives an average width of $\pi/2$.

Show work on these problems

16. Consider the length of the arc of the curve $x^2 = y^3$ going from $(0,0)$ to $(1,1)$. There are three ways to set it up: use x as the variable and think of y as a function of x, use y as the variable and think of x as a function of y, or put in a parameter t going from $t = 0$ to $t = 1$ and set $x = t^3$, $y = t^2$.

    (a) Write down the integrals resulting from any two of these three approaches. The three integrals are all integrals based on the idea that in a right triangle, the length of the hypotenuse is the square root of the sum of the squares of the lengths of the sides. Symbolically, $c = \sqrt{a^2 + b^2}$. Now, if we think of x as the variable, and the curve as made up of a string of tiny hypotenuses strung together, then our a is dx, our b is dy, and our c is the contribution to the arc length, $\sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + (dy/dx)^2}\,dx$. We need dy/dx, and limits of integration. The curve goes from $(0,0)$ to $(1,1)$, so our integral will run from $x = 0$ to $x = 1$. We can solve for y in terms of x; we have $y = x^{2/3}$. Now that's a case of $y = x^n$, with $n = 2/3$, so the derivative is $nx^{n-1} = (2/3)x^{-1/3}$. Slotting that in to our hypotenuse scheme gives

    $$s = \int_{x=0}^{1} \sqrt{1 + \left(\tfrac{2}{3}x^{-1/3}\right)^2}\,dx = \int_{x=0}^{1} \sqrt{1 + \left(\tfrac{4}{9}x^{-2/3}\right)}\,dx.$$

    This looks most unpromising! If we think of x as a function of y and proceed along the same lines we just did, we get

    $$s = \int_{y=0}^{1} \sqrt{1 + \left(\tfrac{9}{4}y\right)}\,dy.$$

    Finally, the parametric approach has $x = t^3$, $dx = 3t^2\,dt$, $y = t^2$, $dy = 2t\,dt$ and so our little hypotenuse now has length $\sqrt{(3t^2\,dt)^2 + (2t\,dt)^2} = \sqrt{9t^4 + 4t^2}\,dt$ and so

    $$s = \int_{t=0}^{1} \sqrt{9t^4 + 4t^2}\,dt.$$

    (b) Choose one of these and evaluate it to actually get the arc length. The dy version can be integrated without too much trouble. Substitute $u = 1 + 9y/4$, $du = (9/4)dy$ to get $dy = (4/9)du$ and get

    $$s = \frac{4}{9}\int_{1}^{13/4} \sqrt{u}\,du = \frac{8}{27}\,u^{3/2}\Big|_{1}^{13/4} = \frac{1}{27}\left(13^{3/2} - 8\right).$$

    The parametric integral can also be done by the techniques of this course. Bring the factor t^2 out, where it becomes just t, and you have $\int_{t=0}^{1} \sqrt{9t^2+4}\,dt$. Set $u = 9t^2 + 4$, $du = 18t\,dt$ and you get $(1/18)\int_{u=4}^{13} u^{1/2}\,du$. This gives the same answer as we got with the dy integral.

17. Find a function $y = f(x)$, defined on some interval about zero, so that $f(0) = 2$ and $f(x)$ satisfies the differential equation $dy/dx = e^{-y}$. We want $dy = e^{-y}\,dx$, so $dy/e^{-y} = dx$ and $e^y\,dy = dx$. Integrating this gives $e^y = x + C$. The '$+C$' must be taken into account immediately, before any other algebra. Why? Because the algebra we do may scramble the effect of $+C$.

    So, taking logs gives $y = \ln(x + C)$. We want $y = 2$ when $x = 0$, because "$f(0) = 2$" means that when $x = 0$, $f(x) = 2$. Thus we need $C = e^2$ and the answer is $f(x) = \ln(x + e^2)$.

18. Series methods as a numerical tool.

   (a) The power series for $1/(1+x)$ is $1 - x + x^2 - x^3 + \cdots$. Write this in Σ-notation. $\sum_{n=0}^{\infty}(-1)^n x^n$. A power series is a sum of terms, each with the form (number) times (nonnegative integer power of the variable.)

   (b) The power series for $\ln(1+x)$ is got by integrating. Give the power series for $\ln(1+x)$. So, you get the power series for $\ln(1+x)$ by integrating something. What to integrate? The derivative, of course. That is,

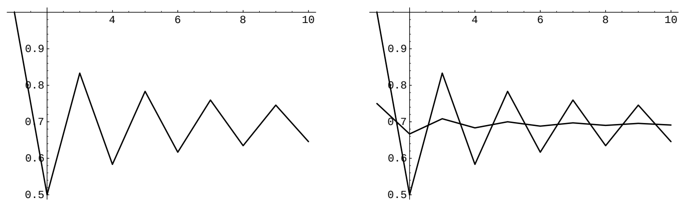
$$\ln(1+x) = \int \frac{dx}{1+x} = \int \sum_{n=0}^{\infty}(-1)^n x^n \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}.$$

   (c) Use the power series for $\ln(1+x)$ to find $\ln(99/100)$, good to ten decimal places. (As in, $1/7 = 0.1428571429$.) Hint: $99/100 = 1 + (-0.01)$. So, we take $x = -0.01$ in this last formula, factor out a minus sign, and conclude that

$$\begin{aligned} -\ln(-0.01) =& 0.010000000000+ \\ & 0.000050000000+ \\ & 0.000000333333+ \\ & 0.000000002500+ \\ & 0.000000000020 + \cdots \\ =& 0.010050335853\cdots \end{aligned}$$

19. On the last hour exam, one of the questions was to give a sensible upper bound for the error in estimating an alternating sum by a partial sum S_n. The answer was that the absolute value of the first term not used, $|a_{n+1}|$, would serve.

   (a) Let $S = \sum_{n=1}^{\infty}(-1)^{n-1}/n = 1 - 1/2 + 1/3 - 1/4 + 1/5\cdots$. Find S_1, S_2, and S_3, and plot them on a line graph by connecting the dots for the points $(1, S_1)$, $(2, S_2)$, and $(3, S_3)$. The essential point to this is to know what the term 'partial sum' means. It's not new, having been a part of the course and part of common exam 3. As indicated by the word 'sum', each S_k is a sum. (Well?) Here, $S_1 = 1$, $S_2 = 1-1/2 = 1/2$, $S_3 = 1-1/2+1/3 = 1/2+1/3 = 3/6+2/6 = 5/6$, and for good measure, $S_4 = 5/6 - 1/4 = 10/12 - 3/12 = 7/12$. Thus, the picture one wants is (I threw in some extra points so you could see how the story plays out) and next to it, the graph for (T_n), superimposed.



(b) Let $T_n = (S_n + S_{n+1})/2$. Find T_1, T_2, and T_3, and draw the graph made of line segments connecting $(1, T_1)$ to $(2, T_2)$ to $(3, T_3)$. Well, let's see.

$$\begin{aligned} T_n =& \frac{1}{2}\left(1 - \frac{1}{2} + \cdots + \frac{(-1)^n}{n}\right) + \\ & \frac{1}{2}\left(1 - \frac{1}{2} + \cdots + \frac{(-1)^n}{n} + \frac{(-1)^{n+1}}{n+1}\right) \\ =& \frac{1}{1}\left(1 - \frac{1}{2} + \cdots + \frac{(-1)^n}{n}\right) + \frac{(-1)^{n+1}}{2(n+1)} \end{aligned}$$

So the T sequence goes 3/4, 2/3, 17/24, and so on. (The plot is above.)

(c) Explain why $\lim_{n\to\infty} T_n = \lim_{n\to\infty} S_n$. For exam purposes, you can, from the wording, take it as given that S_n has a limit, call it L. For exam purposes, this can be put in rough and ready language this way: whenever n is large, S_n is close to L. And of course S_{n+1} is also close to L. So $(S_n + S_{n+1})/2$ is close to $(L + L)/2 = L$. It so happens that the alternating series test applies to S so there really is a limit, and it further happens that from series methods, based on the exam question about $\ln(1 + x)$, give $L = \ln(2) = 0.693147\cdots$.

(d) Find a formula for $T_{n+1} - T_n$, in general. We have

$$\begin{aligned} T_n &= S_n + \frac{1}{2}\frac{(-1)^{n+1}}{n+1} \\ T_{n+1} &= S_n + \frac{(-1)^{n+1}}{n+1} + \frac{1}{2}\frac{(-1)^{n+2}}{n+2} \end{aligned}$$

Subtracting gives

$$T_{n+1} - T_n = \frac{1}{2}(-1)^n\left(\frac{1}{n+2} - \frac{1}{n+1}\right) = \frac{(-1)^{n+1}}{2(n+1)(n+2)}.$$

For exam purposes, because of a weakness in the wording, you could have just written $(S_{n+2} - S_n)/2$. The intent of the question was to ask for a useful formula for $T_{n+1} - T_n$, that would throw some light on the fact, visible from the two zig-zag graphs, that the T_n

sequence oscillates less violently than the S_n sequence. From this algebra, we now know that its oscillations are about the square of the already-going-to-zero oscillations of S_n.

(e) What is the advantage in computing with T_n instead of S_n?The two sequences converge to the same limit, but T_n converges faster. Thus, to get 4 places accuracy from S_n, we would need n to be about 10000, but for T, we could get away with 500 terms.

20. The set of all points u so that the distance from u to $(1, 2, 2)$ is half the distance from u to the origin is a sphere. Find the equation of that sphere. (Hint: first find two points that are both on the sphere, and on the line joining u to the origin.) Bonus: (Extra five points) Prove that this point set really is a sphere, by reducing the vector equation that arises from the definition of the set to the standard form for a sphere.

    Points in three-dimensional space have three coordinates, so our points u have the form $u = (x, y, z)$. We want two points on the line between $(0, 0, 0)$, the origin, and $(1, 2, 2)$, and satisfying the requirement about distance. One such point would be $(2, 4, 4)$, and another, going double the distance from $(0, 0, 0)$ to $(1, 2, 2)$. Another would be $(2/3)(1, 2, 2)$, so that our point is $2/3$ of the way from the origin to $(1, 2, 2)$ which makes it twice as close to $(1, 2, 2)$ as to $(0, 0, 0)$. Now, these two points are the poles of our sphere, and the line segment joining them is a diameter of the sphere, and their midpoint is the center of the sphere. The center of the sphere is thus $(4/3)(1, 2, 2)$, and the radius is the distance from that to $(2, 4, 4)$, which is 2. Specifying the center and radius does the job of specifying the sphere.

    To show that one gets a sphere, observe that the condition on u amounts to this, in vector terms: $|\mathbf{u} - \langle 1, 2, 2\rangle| = \frac{1}{2}|\mathbf{u}|$. Expanding, $4(\mathbf{u} - \langle 1, 2, 2\rangle) \cdot (vvu - \angle 1, 2, 2\rangle) = \mathbf{u} \cdot \mathbf{u}$. Cancelling gives $3\mathbf{u} \cdot \mathbf{u} - 8\mathbf{u} \cdot \langle 1, 2, 2\rangle + 36 = 0$, and this can be put in the form

    $$3(\mathbf{u} - (4/3)\langle 1, 2, 2\rangle) \cdot (\mathbf{u} - (4/3)\langle 1, 2, 2\rangle) = 12$$

    which is the equation of a sphere, because it says that $|\mathbf{u} - (4/3)\langle 1, 2, 2\rangle| = 2$.

    For another solution, write the condition that is required as algebra: the distance from $(1, 2, 2)$ shall be half the distance from $(0, 0, 0)$, translates to

    $$\sqrt{(x-1)^2 + (y-2)^2 + (z-2)^2} = \frac{1}{2}\sqrt{x^2 + y^2 + z^2}.$$

    Square this out and simplify and complete the square to arrive at $3x^2 - 8x + 3y^2 - 16y + 3z^2 - 16z + 36 = 0$ and then $3(x - 4/3)^2 + 3(y - 8/3)^2 + 3(z - 8/3)^2 = 12$. This also proves that the surface in question really is a sphere.