Calculus

Instructions Please write your solutions on your own paper. Explain your reasoning in complete sentences to maximize credit.

1. Suppose f is a function such that $f(0) = 1$ and $f'(0) = 3$. Let g be the composite function such that $g(x) = f(\sin x)$. Determine the value of the derivative $g'(0)$.

   Solution. By the chain rule, $g'(x) = f'(\sin x)\cos x$. Therefore $g'(0) = f'(\sin 0)\cos 0 = f'(0) = 3$.

2. Suppose f is a function such that $f(2) = 4$ and $f'(2) = 7$. Let g be the composite function such that $g(x) = e^{f(2x)}$. Determine the value of the derivative $g'(1)$.

   Solution. By the chain rule, $g'(x) = e^{f(2x)} f'(2x) \times 2$. Therefore $g'(1) = 2e^{f(2)} f'(2) = 14e^4$ (which is approximately equal to 764.37).

3. Suppose f and g are inverse functions [recall this means that $f(g(x)) = x$ and $g(f(x)) = x$], and suppose $f(1) = 2$, $f'(1) = 3$, $f(2) = 5$, and $f'(2) = 7$. Determine the value of the derivative $g'(2)$.

   Solution. According to the rule for the derivative of an inverse function, $g'(2) = 1/f'(1) = 1/3$.

   You could also work out the answer from the chain rule, starting from $g(f(x)) = x$. Differentiating gives $g'(f(x))f'(x) = 1$, so $g'(f(1))f'(1) = 1$, or $g'(2) = 1/f'(1) = 1/3$.

4. According to the TI-89 calculator, the functions $\ln(x)$ and $\ln(171x)$ have the same derivative: namely, $1/x$. Does it make sense that the graphs of these two different functions have the same slope? Explain.

   Solution. One of the basic properties of logarithms tells us that $\ln(171x) = \ln(171) + \ln(x)$. Thus the graph of $\ln(171x)$ is the same as the graph of $\ln(x)$ except translated upward by the fixed amount $\ln(171)$. Translating the tangent lines upward does not change their slope. To put it another way, the two functions have the same derivative because they differ by a constant (and the derivative of a constant function is 0).

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5. Suppose $f(2) = 3$ and $f'(2) = 5$. Let g be the function such that $g(x) = x^{f(x)}$. Use logarithmic differentiation to determine the value of the derivative $g'(2)$.

 Solution. Since $\ln g(x) = f(x) \ln x$, differentiating (using the chain rule on the left-hand side and the product rule on the right-hand side) shows that $g'(x)/g(x) = f'(x) \ln x + f(x)/x$. Consequently, $g'(2)/g(2) = f'(2) \ln 2 + f(2)/2$, or $g'(2) = 2^3(5 \ln 2 + 3/2) = 40 \ln 2 + 12$ (which is approximately equal to 39.7).

6. Show that the curves $y = x^3$ and $x^2 + 3y^2 = 1$ intersect orthogonally.

 Solution. For the first curve, the slope is $y' = 3x^2 = 3y/x$. For the second curve, implicit differentiation shows that $2x + 6yy' = 0$, or $y' = -x/(3y)$. The product of the two slopes is equal to -1, which means that the curves intersect at right angles.

 It is not necessary to know the intersection points, which are approximately $\pm(0.7325, 0.393)$.

7. Two students leave Milner Hall at the same time, walking in perpendicular directions. One student walks northeast along Ross Street at a speed of 3 feet per second, and the other student walks northwest along Asbury Street at a speed of 4 feet per second. At what rate is the distance between the students increasing 10 seconds after they start walking?

 Solution. Let $x(t)$ denote the distance that the first student has walked by time t, let $y(t)$ denote the distance that the second student has walked by time t, and let $D(t)$ denote the distance between the students at time t. We are given that $dx/dt = 3$ and $dy/dt = 4$, and we are supposed to determine dD/dt when $t = 10$. The relation between the variables is $D^2 = x^2 + y^2$ (by the Pythagorean law for right triangles).

 Differentiating shows that $2D(dD/dt) = 2x(dx/dt) + 2y(dy/dt)$, or $dD/dt = (3x + 4y)/D$. Now $x = 3t$ and $y = 4t$, so $D = 5t$. Therefore $dD/dt = (9t + 16t)/(5t) = 5$, so the rate of change of the distance D with respect to time is 5 feet per second. This rate is independent of t, so we do not need to use the information that $t = 10$.

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8. Consider the curve given in parametric form by $x(t) = t^3$ and $y(t) = \sin(t)$, where t runs over the real numbers. Do these parametric equations determine y as a one-to-one function of x? Explain why or why not.

   Solution. The parametric equations determine y as a function of x, but it is not a one-to-one function. For example, if $t = 0$ then $x = 0$ and $y = 0$, while if $t = \pi$ then $x = \pi^3$ and $y = 0$. Since the same value of y corresponds to two values of x, the function is not one-to-one.

9. If you invest \$1,000 at 5% interest compounded continuously, then the amount $A(t)$ in your account after t years is $A(t) = 1{,}000\, e^{t/20}$. How many years does it take to double your money?

   Solution. We need to solve the equation $2{,}000 = 1{,}000\, e^{t/20}$ for t. Dividing by 1,000 shows that $2 = e^{t/20}$, so $\ln 2 = t/20$, or $t = 20 \ln 2$ (which is approximately 13.86 years).

10. **Optional problem for extra credit**
    Jackie finds an approximate value for $\sqrt[3]{1001}$ (the cube root of 1001) by using the linear approximation for the function $\sqrt[3]{x}$ with the base point $a = 1000$. Jamie finds an approximate value for $\sqrt[3]{1001}$ by doing one iteration of Newton's method applied to the function $x^3 - 1001$ with starting point $x_0 = 10$. Whose approximation to $\sqrt[3]{1001}$ is more accurate?

    Solution. Jackie's calculation, using $f(x) = \sqrt[3]{x}$ and $f'(x) = 1/(3x^{2/3})$ and the linear approximation formula $f(x) \approx f(a) + f'(a)(x - a)$, is

    $$\sqrt[3]{1001} \approx \sqrt[3]{1000} + \frac{1}{3(1000)^{2/3}}(1001 - 1000) = 10 + \frac{1}{300}.$$

    Jamie's calculation, using $f(x) = x^3 - 1001$ and $f'(x) = 3x^2$ and Newton's formula $x_1 = x_0 - f(x_0)/f'(x_0)$, is

    $$x_1 = 10 - \frac{1000 - 1001}{3(10^2)} = 10 + \frac{1}{300}.$$

    The two approximations are equal. (The equality is no coincidence, for both methods use a tangent line approximation: one for a function and one for the inverse function.)