Linear Algebra

Instructions Please use complete sentences, along with any necessary supporting calculations, to answer the following questions.

1. Consider the system
$$\begin{cases} 2x_1 + x_2 = a^2 \\ 6x_1 + 3x_2 = a \end{cases}$$
of simultaneous equations for the unknowns x_1 and x_2, where a is a certain constant. For which value(s) of the constant a is the system of equations *consistent*? How do you know?

Solution. One way to answer the question is to attempt to solve the system of equations and to see what could go wrong.

Subtracting 3 times the first equation from the second equation gives the equivalent system

$$\begin{cases} 2x_1 + x_2 = a^2 \\ 0x_1 + 0x_2 = a - 3a^2. \end{cases}$$

The new second equation is impossible unless $a - 3a^2 = 0$. Therefore consistency requires that either $a = 0$ or $1 - 3a = 0$. Thus the values of a for which the system is consistent are 0 and 1/3.

Remarks

- You can find *one* of the special values of a without doing any calculation. When $a = 0$, the system is homogenous (zero right-hand side). It is an important general principle that a homogeneous system is always consistent, because there is the "trivial solution" $x_1 = x_2 = 0$.
- This problem is one where you had better think before reaching for a calculator. The command

  ```
  rref([2,1,a^2;6,3,a])
  ```

  returns the result
  $$\begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

both on the TI-89 calculator and in MATLAB, apparently showing that the system is always inconsistent. The calculator and the computer both miss the two special values of a for which the system is consistent.

- Another way to analyze the problem is to use the Consistency Theorem (Theorem 1.3.1) that we saw in class today. The system can be rewritten in the form

$$x_1 \begin{pmatrix} 2 \\ 6 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} a^2 \\ a \end{pmatrix}.$$

Thus the system is consistent precisely when the column vector $\begin{pmatrix} a^2 \\ a \end{pmatrix}$ can be written as a linear combination of the column vectors $\begin{pmatrix} 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Since the latter two vectors are proportional, the system is consistent precisely when the colum vector $\begin{pmatrix} a^2 \\ a \end{pmatrix}$ is a multiple of the column vector $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$. In other words, the second component of the vector is 3 times the first component, which shows that $a = 3a^2$. Solving this equation for a, we find again that either $a = 0$ or $a = 1/3$.

Linear Algebra

2. Rose is studying the linear system

$$\begin{aligned} x_1 + \ \ 2x_2 + \ \ 3x_3 &= \ \ 4 \\ 5x_1 + \ \ 6x_2 + \ \ 7x_3 &= \ \ 8 \\ 9x_1 + 10x_2 + 11x_3 &= 12 \end{aligned} \tag{$\dagger$}$$

of three equations in the three unknowns x_1, x_2, and x_3. Rose discovers that the TI-89 calculator has a command `rref` (which stands for "reduced row echelon form"), and the command

```
rref([1,2,3,4;5,6,7,8;9,10,11,12])
```

returns the output

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

What should Rose conclude about the set of solutions of the linear system (†)?

Solution. There is no single "right answer" to this question. Any one of the following deductions is a reasonable answer.

- The linear system has infinitely many solutions.
- Viewed geometrically, the solution set is a line in three-dimensional space.
- The lead variables x_1 and x_2 can be determined in terms of the free variable x_3.
- The solution of the linear system can be written as follows:

$$\begin{aligned} x_1 &= -2 + x_3 \\ x_2 &= 3 - 2x_3 \\ x_3 &\text{ is arbitrary.} \end{aligned}$$