Linear Algebra

Instructions Please write your name in the upper right-hand corner of the page. Use complete sentences, along with any necessary supporting calculations, to answer the following questions.

1. Find a 2×2 matrix A that has the vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}2\\1\end{pmatrix}$ as eigenvectors with corresponding eigenvalues 3 and 4.

   Solution. This problem can be solved in at least two ways.

   Method 1: solution from first principles. If $A = \begin{pmatrix}a & b\\c & d\end{pmatrix}$, then the given information tells us that

   $$\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}3\\0\end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}8\\4\end{pmatrix}.$$

   The first equation says that

   $$\begin{pmatrix}a\\c\end{pmatrix} = \begin{pmatrix}3\\0\end{pmatrix}.$$

   The second equation says that

   $$2\begin{pmatrix}a\\c\end{pmatrix} + \begin{pmatrix}b\\d\end{pmatrix} = \begin{pmatrix}8\\4\end{pmatrix}, \qquad \text{or} \qquad \begin{pmatrix}6\\0\end{pmatrix} + \begin{pmatrix}b\\d\end{pmatrix} = \begin{pmatrix}8\\4\end{pmatrix},$$

   so $\begin{pmatrix}b\\d\end{pmatrix} = \begin{pmatrix}2\\4\end{pmatrix}$. Hence $A = \begin{pmatrix}3 & 2\\0 & 4\end{pmatrix}$.

   Method 2: the thematic approach. Multiplication by the matrix A defines a linear operator on R^2. This operator is represented in the eigenvector basis by the diagonal matrix $D = \begin{pmatrix}3 & 0\\0 & 4\end{pmatrix}$. The transition matrix S from the eigenvector basis to the standard basis has the eigenvectors as its columns:

   $$S = \begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}.$$

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Now $S^{-1}AS = D$, so

$$A = SDS^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 0 & 4 \end{pmatrix}.$$

2. If a 2×2 matrix A has the numbers 1 and 3 as eigenvalues, and a 2×2 matrix B has the numbers 1 and 4 as eigenvalues, must the product matrix AB have the numbers 1 and 12 as eigenvalues? Explain why or why not.

   Solution. The statement will hold if the matrices A and B have the same eigenvectors, but there is no reason to expect the product AB to be special if the eigenvectors of A and B do not match. Here are some examples showing that AB can have eigenvalues different from 1 and 12.

   If $A = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$, then A and B have the required eigenvalues, but

   $$AB = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix},$$

   which has eigenvalues 4 and 3.

   If $A = \begin{pmatrix} 1 & 5 \\ 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ -1 & 4 \end{pmatrix}$, then A and B have the required eigenvalues, but

   $$AB = \begin{pmatrix} -4 & 20 \\ -3 & 12 \end{pmatrix},$$

   which has eigenvalues 2 and 6.

   Recall that the product of the eigenvalues of a matrix equals the determinant, and the sum of the eigenvalues equals the trace. If $A = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & -6 \\ 3 & 7 \end{pmatrix}$, then A and B have the required eigenvalues (notice that $\det(B) = 4$ and $\operatorname{trace}(B) = 5$), but

   $$AB = \begin{pmatrix} -2 & -6 \\ 9 & 21 \end{pmatrix},$$

   and $\operatorname{trace}(AB) = 19 \neq 13$, so AB does not have eigenvalues 1 and 12.