

## Complex Variables

**Instructions** Please write your solutions on your own paper. Explain your reasoning in complete sentences to maximize credit.

1. Either sketch or describe in words the set of non-zero complex numbers  $z$  for which  $\operatorname{Re}(i/z) = 0$ .

**Solution.** If  $z = x + iy$ , then

$$\frac{i}{z} = \frac{i}{x + iy} = \frac{i(x - iy)}{(x + iy)(x - iy)} = \frac{y + ix}{x^2 + y^2},$$

so  $\operatorname{Re}(i/z) = y/(x^2 + y^2)$ . Consequently, if  $z \neq 0$ , then  $\operatorname{Re}(i/z) = 0$  if and only if  $y = 0$ . The set of all such complex numbers is the real axis with the point 0 removed.

2. Determine (in the standard form  $x + iy$ ) all complex numbers  $z$  such that  $z^4 + 4 = 0$ .

**Solution.** The question asks for the fourth roots of the number  $-4$ . Writing  $z$  in polar form  $re^{i\theta}$  shows that  $r^4 e^{4i\theta} = 4e^{i\pi}$ . Therefore  $r = 4^{1/4} = \sqrt{2}$ . The possible values of the argument  $\theta$  are  $\pi/4, 3\pi/4, 5\pi/4$ , and  $7\pi/4$ . Now  $e^{i\pi/4} = (1 + i)/\sqrt{2}$ , so one value of  $z$  is  $1 + i$ . The other three values, symmetrically located, are  $-1 + i, -1 - i$ , and  $1 - i$ .

3. Explain how you know that the “punctured plane” consisting of all non-zero complex numbers is both an open set and a connected set.

**Solution.** If  $z$  is a non-zero complex number, then the open disc with center  $z$  and radius  $|z|$  is contained in the punctured plane. Thus the punctured plane is an open set, because it contains a disc centered at an arbitrary point of the set.

The punctured plane is connected because every two points in the set can be joined by a polygonal path contained in the set. Indeed, if the two points are not on the same line through the origin, then the line segment that joins the points is such a path. If two points are on the same line through the origin, then join the first point by a line segment to some point not on the line, and then join that point by a line segment to the second point.

## Complex Variables

4. Does the infinite series  $\sum_{n=2}^{\infty} \frac{1}{2-i^n}$  converge or diverge? Explain how you know.

**Solution.** By the triangle inequality,  $|2 - i^n| \leq 2 + |i^n| = 3$ , so  $|1/(2 - i^n)| \geq 1/3$ . Since the terms of the series do not tend to 0, the series diverges.

Alternatively, one could argue that the terms of the series repeat in blocks of four, so after adding  $k$  blocks one has a partial sum that equals  $k$  times the sum of the first four terms. Evidently, then, the partial sums are unbounded, so the series diverges.

5. Find all (infinitely many) values of the expression  $i^{(2+i)}$ . Write the values in the standard form  $a + bi$ .

**Solution.** By definition,  $i^{2+i} = e^{(2+i)\log(i)}$ . Now  $\log(i) = \ln|i| + i \arg(i) = 0 + i(\frac{\pi}{2} + 2\pi n)$ , where  $n$  is any integer. Therefore

$$i^{2+i} = e^{(2+i)i(\frac{\pi}{2}+2\pi n)} = e^{\pi i - \frac{\pi}{2} + 4\pi i n - 2\pi n} = -e^{-\frac{\pi}{2} - 2\pi n},$$

where  $n$  is any integer. It is amusing to note that the values of  $i^{2+i}$  happen all to be real numbers.

6. Evaluate the line integral  $\int_{\gamma} \frac{1}{z} dz$ , where  $\gamma$  is a quarter circle in the first quadrant joining the point 1 to the point  $i$ . Write the answer in the standard form  $a + bi$ .

**Solution.** If  $\gamma$  is parametrized by  $e^{i\theta}$ , then the integral becomes

$$\int_0^{\pi/2} \frac{1}{e^{i\theta}} i e^{i\theta} d\theta = \frac{i\pi}{2}.$$

Actually, the statement of the problem is ambiguous, because there is another quarter circle in the first quadrant that joins 1 to  $i$ : namely, a quarter of the circle with center at  $1+i$  and radius 1. The integrand  $1/z$

## Complex Variables

is analytic in the first quadrant, however, so the integral is independent of the path in the first quadrant.

Since the value of this integral depends only on the endpoints of  $\gamma$ , one might alternatively argue as follows: an anti-derivative of  $1/z$  is  $\text{Log}(z)$ , so the answer should be  $\text{Log}(i) - \text{Log}(1)$ , or  $\ln|i| + i \text{Arg}(i) - 0$ , or  $i\pi/2$ . This argument is correct, although we do not officially know that yet.

7. Suppose  $f(z) = |z|^2$ . Show that the (complex) derivative  $f'(z)$  does not exist unless  $z = 0$ .

**Solution.** If  $f = u + iv$ , then  $u(x, y) = x^2 + y^2$  and  $v(x, y) = 0$ . The partial derivatives of  $v$  are equal to 0, so if  $f$  were complex differentiable, then the Cauchy-Riemann equations would imply that  $0 = \partial u/\partial x = 2x$  and  $0 = \partial u/\partial y = 2y$ . Thus the Cauchy-Riemann equations do not hold unless  $x = y = 0$ , so the function  $f$  cannot be complex differentiable unless  $z = 0$ .

Alternatively, one could argue from the definition of the derivative as a limit:

$$f'(z) = \lim_{h \rightarrow 0} \frac{|z+h|^2 - |z|^2}{h} = \lim_{h \rightarrow 0} \frac{\bar{h}z + \bar{z}h + \bar{h}h}{h} = \bar{z} + z \lim_{h \rightarrow 0} \frac{\bar{h}}{h}.$$

Now  $\bar{h}/h$  has the value 1 when  $h$  is real and the value  $-1$  when  $h$  is purely imaginary, so  $\lim_{h \rightarrow 0} (\bar{h}/h)$  does not exist as a two-dimensional limit. Therefore the limit defining  $f'(z)$  does not exist unless  $z = 0$ .

8. Determine the radius of convergence of the power series  $\sum_{n=1}^{\infty} \frac{(-1)^n}{4^n} z^{2n}$ .

**Solution.** The series equals the geometric series  $\sum_{n=1}^{\infty} (-z^2/4)^n$ , so the series converges precisely when  $|-z^2/4| < 1$ , or when  $|z| < 2$ . Thus the radius of convergence equals 2.

Alternatively, one could argue by the root test that the series converges (absolutely) when  $\lim_{n \rightarrow \infty} |(-1)^n z^{2n}/4^n|^{1/n} < 1$ . That limit evidently equals  $|z^2/4|$ , so we see again that the radius of convergence equals 2. One could also use the ratio test to get the same answer.