SETS AND SET NOTATION

A SET is a collection of items.

The items are called the MEMBERS or ELEMENTS of the set.

A set is given a name, usually an uppercase letter.

We can use SET BUILDER notation to describe a set in terms of its properties,

$A = \{x|x \text{ is a female in math 166 this semester}\}$.

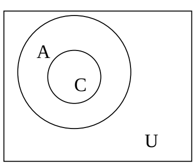
A UNIVERSAL SET is a set from which all the member of the sets in a problem can be drawn.

$U = \{x|x \text{ is a student in math 166 this semester}\}$

We use VENN DIAGRAMS to show sets. The rectangle is the universal set and the circles are sets in the universal set.

A set C is a subset of A if every element in C is also in A. Define

$C = \{x|x \text{ is a female GEST major in math 166 this semester}\}$,

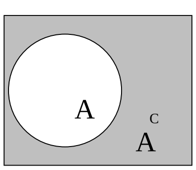
$C \subseteq A$ 

If $B = \{x|x \text{ is a GEST major in Math 166 this semester}\}$, then B i s not a subset of A

A set with no members is the EMPTY SET. The notation for the empty set is $\emptyset$ or $\{\}$. Do not confuse the empty set with the real number 0, or the set with the real number zero as an element, $\{0\}$.

The COMPLEMENT of set A are those elements of U that are NOT in A. We write this as

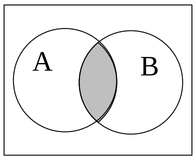
$A^c = \{x|x \in U \text{ and } x \notin A\}$. Shade the region A^c.



The INTERSECTION of sets A and B are those elements that are in A AND in B. In set notation we say $A \cap B$. In set builder notation we say

$\{x | x \in A \textit{ and } x \in B\}$.

Shade the region $A \cap B$.

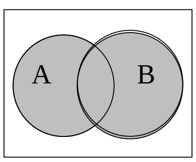


In our example, $A \cap B$ would be female GEST majors in math 166 this semester.

The UNION of sets A and B are those elements that are in A OR in B. In set notation we say $A \cup B$. In set builder notation

$\{x | x \in A \textit{ and } x \in B\}$.

Shade the region $A \cup B$:



In our example, $A \cup B$ would be the students who are in math 166 semester who are GEST majors or are female.

You must be able to find and express any part of a Venn diagram using set notation, set builder notation or words.

Example - Using our sets from above,

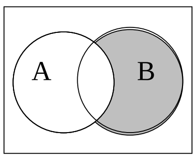
$U = \{x|x \text{ is a student in math 166 this semester}\}$

$A = \{x|x \text{ is a female in math 166 this semester}\}$

$B = \{x|x \text{ is a GEST major in math 166 this semester}\}$

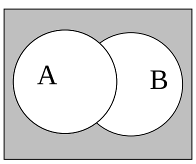
Shade the following regions and express in set notation:

a male who is a GEST major in math 166 this semester



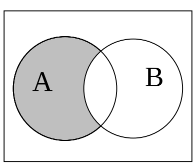
$B \cap A^c$

a male in math 166 this semester who is not a GEST major.



$(A \cup B)^c$ or $A^c \cap B^c$

a female who is in math 166 ths semester and is not a GEST major.



$A \cap B^c$

A three circle Venn diagram has 8 different regions.

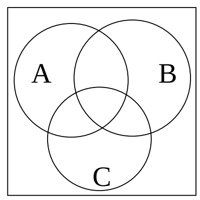
Example:

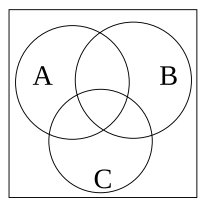
There are pizzas available at a buffet. Define the following

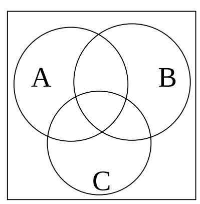
$A = \{x | x \text{ has sausage on the pizza}\}$

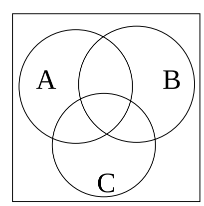
$B = \{x | x \text{ has pepperoni on the pizza}\}$

$C = \{x | x \text{ has mushrooms on the pizza}\}$

a) Find the region for only sausage, $A \cap B^c \cap C^c$ 

b) Find the region for mushrooms and pepperoni, $B \cap C$ 

c) Find the region for not sausage, A^c. 

d) Find the region for only one ingredient, $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap)$ 

NUMBER OF ELEMENTS IN A FINITE SET

We use the notation $n(A)$ for the number of elements in set A. It is important to not double count elements in a Venn diagram. For two sets we have the

UNION RULE:

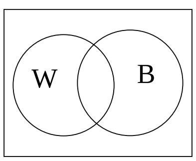
$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Example: In a class of 100 students we have 50 women and 60 GEST majors. If 80 of the students are women or GEST majors, how many students are

a) women GEST majors?

b) men GEST majors?

c) men non-GEST major?



We need the center number, the intersection.

Use the union rule,

$$n(W \cup B) = n(W) + n(B) - n(W \cap B) \to 80$$

$$= 50 + 60 - n(W \cap B)$$

$$\to n(W \cap B) = 110 - 80 = 30$$

Fill in the intersection. Find the rest of the numbers with subtraction. Now with a complete diagram we can answer the questions.

Example - There were 150 dogs at a dog show.

Let A be the set of big dogs (over 50 pounds).

Let B be the set of dogs with long hair.

Let C be the set of dogs with spots.

95 dogs had spots

8 dogs had spots and long hair and were big.

12 dogs had only long hair

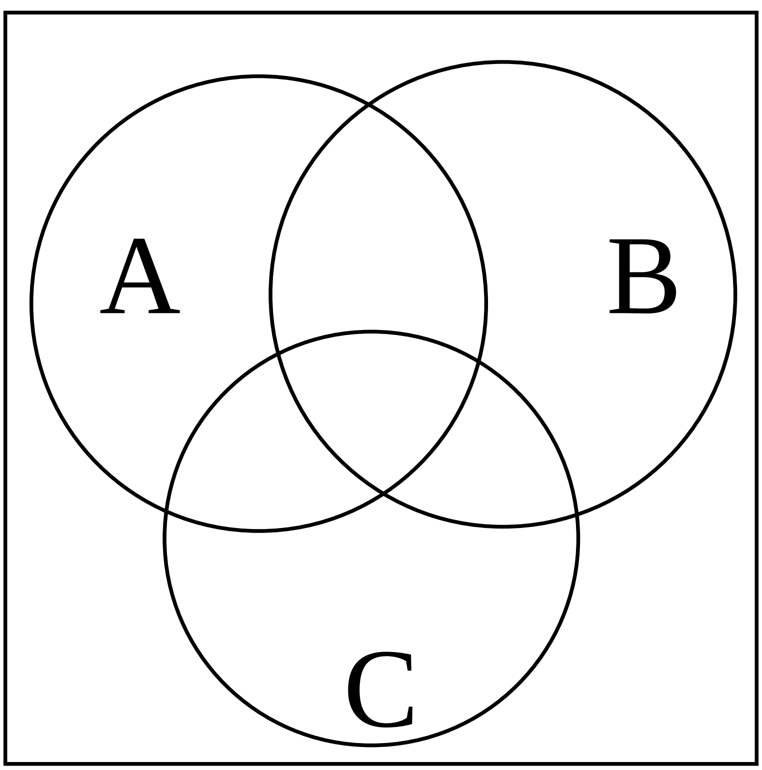
25 dogs had long hair and were big

23 dogs had only spots and were big

92 dogs were not big

49 had exactly two of these characteristics.

Complete the Venn diagram for the above information:



THE MULTIPLICATION PRINCIPLE

At a pasta diner there is a choice of 4 different pastas and 3 different sauces. How many dinners can be made? Use a TREE DIAGRAM to organize the choices and outcomes

We find 12 different dinners and $12 = 4 \cdot 3$. What if there were 5 different meats to choose from? We need the MULTIPLICATION PRINCIPLE:

Suppose a task T_1 can be completed n_1 ways, a task T_2 can be completed n_2 ways, $\ldots$ and a task T_r can be completed n_r ways. The number outcomes from making these r choices is the product

$$n_1 \cdot n_2 \cdot \ldots n_r$$

So there would be $4 \cdot 3 \cdot 5 = 60$ possible dinners with the additonal choice of 5 meats.

Example - How many different computer addresses are possible if the first three spots have letters and the last four spots have digits?

We have 26 ways to complete the task of choosing a letter for each of the letter places and 10 ways to complete the task of choosing a digit for each of the four digit places. In all

$$26 \cdot \ 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 175,760,000$$

Example - How many different 4-digit access codes can be made if the first digit cannot be a 0 or a 1 and no repeats are allowed?

The first spot has 8 choices available. Now one digit is removed from our possible digit list since we cannot have repeats, but now the 0 and 1 are available and so the next spot has 9 choices, then 8 then 7,

$8 \cdot 9 \cdot 8 \cdot 7 = 4032$

Example - How many different 2 scoop ice cream cones are possible if there are 31 flavors to choose from?

31 choices for the first scoop and 31 choices for the second scoop = $31 \cdot 31 = 961$ different two scoop cones.

PERMUTATIONS

A common application of the multiplication principle is to choose elements from a finite set and arrange them in a certain way.

How many ways can 8 different books be arranged on a bookshelf?

We have 8 choices available for the first spot on the shelf. Choose a book out and place it on the shelf. Now we have 7 left to choose from for the second spot, etc...

$$8 \cdot 7 \cdot 6 \cdot 5 \cdot 5 \cdot 3 \cdot 2 \cdot 1 = 8! = 40,320$$

We find $8!$ (on the calculator, enter 8, then MATH button. Go to the PRB menu. Enter 4 or scroll down to the ! sign. Then enter again for the answer)

But this is also called a PERMUTATION, $P(8,8)$, the permutation of 8 things taken 8 at a time.

What if we arranged only 3 of the books? How many ways to do this?

$8 \cdot 7 \cdot 6 = 336 = \frac{8!}{5!} = \frac{8!}{(8-3)!}$

This is a permutation of 8 things taken 3 at a time, or $P(8,3)$. You can find these on your calculator as $8nPr3$. (enter 8 on the home screen. hit the MATH button then choose the PRB menu. enter 2 or scroll down to nPr and enter. then hit 3 then enter again)

The general formula for the permutation of n things taken r at a time is

$$P(n,r) = \frac{n!}{(n-r)!}$$

If some of the objects being arranged are the same, then we have to find those permutations that look different, DISTINGUISHABLE PERMUTATIONS.

Say we have a total of 6 marbles. 3 of them are blue, 2 of them are green and 1 is red. How many different distinguishable permutations are there?

The formula for the number of distinguishable permutations of N items where n_1 items are of type 1 and n_2 items are of type 2 and n_r items are of type r is

$$\frac{N!}{n_1!n_2!\ldots n_r!}$$

So in our example we would have

$$\frac{6!}{3!2!1!} = 60$$

We often only want a group or subset of items from a finite set, not an arrangement. When the order of the objects doesn't matter, it is a COMBINATION.

Say we wanted to choose 3 of 8 books to lend to a friend. The order they are chosen won't matter, just if the books are chosen to be in the group or not. We start by finding the number of ways to arrange the 3 of 8 books and then divide by the number of ways the 3 books can be arranged among themselves (as they are all still in the same group). We have

$$\frac{8 \cdot 7 \cdot 6}{3!} = 56 = C(8,3) = \frac{P(8,3)}{3!} = \binom{8}{3}$$

In general, the COMBINATION of n things taken r at at time is

$$C(n,r) = \binom{n}{r} = \frac{n!}{(n-r)!r!}$$

Example - A researcher has 12 plants. 5 of them are wheat, 4 are corn and 3 are rye plants. A sample of 3 plants in chosen.

a) How many different samples of 3 plants are there?

We have 12 plants and we pick a group of 3 to be our sample,

$$C(12,3) = \binom{12}{3} = 12nCr3 = 220$$

b) How many different samples have 2 wheat plants?

We will choose 2 from the 5 wheats and the remaining 1 from the 7 that are not wheat:

$$C(5,2) \cdot C(7,1) = \binom{5}{2} \cdot \binom{7}{1}$$

$$= (5nCr2) \cdot (7nCr1) = 10 \cdot 7 = 70$$

c) How many different samples will have one of each kind?

$$C(5,1) \cdot C(4,1) \cdot C(3,1) = \binom{5}{1} \cdot \binom{4}{1} \cdot \binom{3}{1}$$

$$= (5nCr1) \cdot (4nCr1) \cdot (3nCr1) = 5 \cdot 4 \cdot 3 = 60$$

d) How many samples will have at least one corn?

First find which samples will satisfy our "at least" statement:

1 corn and 2 not corn + 2 corn and 1 not corn + 3 corn and 0 not corn

Now work out how many ways to make each sample and then add them all up:

1 corn and 2 not corn is

$$C(4,1) \cdot C(8,2) = \binom{4}{1} \cdot \binom{8}{2} = (4nCr1) \cdot (8nCr2) = 4 \cdot 28 = 112$$

2 corn and 1 not corn is

$$C(4,2) \cdot C(8,1) = \binom{4}{2} \cdot \binom{8}{1} = (4nCr2) \cdot (8nCr1) = 6 \cdot 8 = 48$$

3 corn and 1 not corn is

$$C(4,3) \cdot C(8,0) = \binom{4}{3} \cdot \binom{8}{0} = (4nCr3) \cdot (8nCr0) = 4 \cdot 1 = 4$$

So we will have

$$112 + 48 + 4 = 164$$

ways to choose a sample with at least one corn.

An alternative way to solve this problem is to realize that in any sample of 3 plants we can have $3C, 0C^c$ or $2C, 1C^c$ or $1C, 2C^c$ or $0C, 3C^c$. So we can find the number of ways to choose without restrictions (220) and subract our one missing case, $0C, 3C^c$,

$$C(4,0) \cdot C(8,3) = \binom{4}{0} \cdot \binom{8}{3} = (4nCr0) \cdot (8nCr3) = 1 \cdot 56 = 56$$

And so find the number of ways to have at least one corn is 220 - 56 = 164, same as calculating the three cases that work. This is is a useful technique for "at least one" type problems.

Example - How many ways can a hand of 5 cards have three cards of the same suit?

Example - In a bag of 10 apples there are 3 bad, apples.

How many ways can a sample of 4 be chosen?

How many ways in which

a) None are rotten?

b) 1 is rotten?

c) 2 are rotten?

d) 3 are rotten?

e) 4 are rotten?

Example - A minivan can hold 7 passengers. An adult must sit in one of the two front seats and anyone can sit in the rear 5 seats. A group of 4 adults and 3 children are to be seated in the van. How many different seating arrangements are possible?

Example - How many different two item pizzas are possible if there are 8 different toppings and doubles are allowed?

Example - You have a class of 20 children, 10 boys and 10 girls. How many ways can the children be seated in a row if boys and girls must alternate?

Example - You take a mutiple choice test with 3 questions and each question has 5 possible answers. How many ways can the test be answered?