

## Conditional Probability

We are given the following data about a basket of fruit:

	rotten	not rotten	total
apple	3	6	9
orange	2	4	6
total	5	10	15

We can find the probability that a fruit is

an apple,  $P(A) = 9/15$

an orange,  $P(O) = 6/15$

rotten,  $P(R) = 5/15$

a rotten apple,  $P(A \cap R) = 3/15$

a not rotten orange,  $P(O \cap R^c) = 4/15$

What is the probability that an apple is rotten? There are 9 apples and 3 of them are rotten, so

$$P(R|A) = 3/9$$

NOTATION:  $P(E|F) =$  "the probability of  $E$  given  $F$ "

We have reduced the sample space by knowing that the fruit was an apple. We have a formula for this:

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

Using the formula,

$$P(R|A) = \frac{P(R \cap A)}{P(A)} = \frac{3/15}{9/15} = \frac{3}{9}$$

What is the probability that a rotten piece of fruit is an orange?

$$P(O|R) = \frac{P(O \cap R)}{P(R)} = \frac{2/15}{5/15} = \frac{2}{5}$$

We choose a marble from a cup or a bowl. The probability of choosing the cup is  $1/4$ . The cup contains 3 purple and 2 green marbles. The bowl contains 2 purple and 4 green marbles. What is the probability that:

- a) a purple marble is chosen from the cup?
- b) a green marble is chosen?

We must make a tree diagram to show this process of first choosing the container and then choosing a marble from the container.

How can we find  $P(P \cap C)$ ? Use our rule for the conditional probability

$$P(E|F) = \frac{P(E \cap F)}{P(F)} \rightarrow P(E \cap F) = P(F)P(E|F)$$

this is called the PRODUCT RULE. It means to multiply along the branch to get the total probability.

a)  $P(P \cap C) = P(C)P(P|C) = \frac{1}{4} \cdot \frac{3}{5} = \frac{3}{20}$

b) You can pick a green marble from the cup or from the bowl, so

$$P(G) = P(G \cap C) + P(G \cap B) = \frac{1}{10} + \frac{1}{2} = \frac{3}{5}$$

This was a stochastic process - what happened depended on what happened earlier in the experiment. The tree diagram shows this as a process: first pick the container, then a marble. Another stochastic process is to choose cards from a deck and NOT replace them.

What is the probability that the third card drawn from a standard deck is a face card given the first two cards were face cards? What is the probability of drawing three face cards?

Make a tree diagram to show this process. We don't need the whole tree, just the part that will give us our answers:

### Independent Events

Events can be independent of each other - when you flip two coins, knowing the first coin was a head tells you nothing about the second coin. When two events  $E$  and  $F$  are independent then

$$P(E \cap F) = P(E) \cdot P(F)$$

We will use this equation in two ways. First, if we know that two events are independent, we can multiply the probabilities to find the probability that both happen. Second, we can use the equation to TEST if two events are independent.

Example - there are 3 laser printers available. Printer A has a 5% chance of jamming, printer B has a 2% chance of jamming and printer C has a 10% chance of jamming. What is the probability that they will all jam?

$$\begin{aligned} P(A \cap B \cap C) &= P(A) \cdot P(B) \cdot P(C) \\ &= .05 \cdot .02 \cdot .10 = .001 = .1\% \end{aligned}$$

Example - Using the Venn diagram, find if  $A$  and  $E$  are independent:

### Baye's Theorem

Look back at our marble choosing problem, what is the probability a marble came from the cup given that it is green,  $P(C|G)$ ? This is NOT on the diagram, we will have to use the formula,

$$P(C|G) = \frac{P(C \cap G)}{P(G)} = \frac{1/10}{1/10 + 1/2} = \frac{1}{6}$$

This is Baye's theorem. The denominator in our formula is not given, you must find it as a sum from the ways that the event can happen. What is the probability the marble came from the bowl given it is purple?

$$P(B|P) = \frac{P(B \cap P)}{P(P)} = \frac{1/4}{1/4 + 3/20} = \frac{5}{8}$$

The formulas in the book are quite nasty - you should just understand that it is the SAME formula as the last section, just now there is a sum for the denominator. We can have more than two things in our denominator if more things can happen:

Example - A manufacturer buys items from each of 6 different suppliers. The fraction of the total number of items obtained from each supplier, along with the probability that an item purchased from the supplier is defective is given in the following table:

supplier	fraction of total supplied	probability of defect
A	.14	.04
B	.47	.02
C	.16	.07
D	.23	.01

What is the probability that a defective item came from supplier B,  $P(B|d)$ ?  
Make a tree diagram and find the parts for the formula

$$\begin{aligned}
 P(B|d) &= \frac{P(B \cap d)}{P(d)} \\
 &= \frac{P(B \cap d)}{P(A \cap d) + P(B \cap d) + P(C \cap d) + P(D \cap d)} \\
 &= \frac{.47 \cdot .02}{.14 \cdot .04 + .47 \cdot .02 + .16 \cdot .07 + .23 \cdot .01} \approx .33
 \end{aligned}$$

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

Example - Two cards are drawn without replacement at random from a standard deck of 52 cards. What is the probability that

- a) The second card is red, given the first card is black?
- b) The second card is black, given the first card is black?
- c) The first card was black, given the second card was red?
- d) The first card was red, given the second card was red?
- c) Both cards were red?

Example - A pair of fair dice are rolled and the uppermost dots are noted. What is the probability that

- a) the sum is greater than 10 given that at least one of the die shows a 6?
- b) the sum is 4 if the roll was a double (both die showing the same number of dots)?
- c) the sum is less than 6 given that the two numbers showing are different (not a double)?