

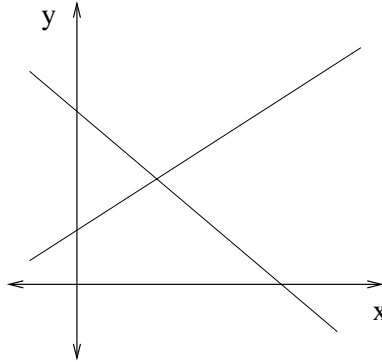
## WEEK 3 REVIEW

- Linear Systems (2.1)
- Systems of Linear Equations I (2.2)
- Systems of Linear Equations II (2.3)
- Matrices (2.4)
- Matrix Multiplication (2.5)
- Matrix Inverses (2.6)

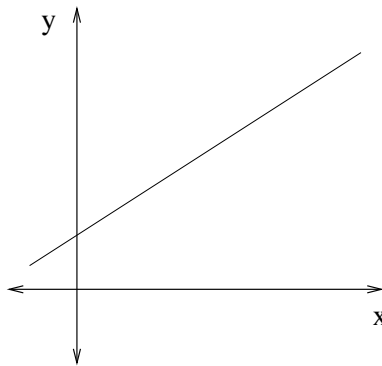
## LINEAR SYSTEMS

When you have a system of two equations and two unknowns we have three possibilities for the two lines:

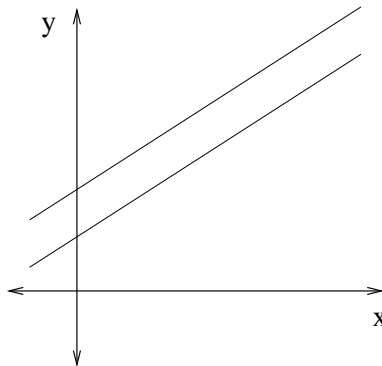
case 1 - The two lines intersect. Solution is the single point of intersection.



case 2 - The two lines are the same. The solution is the entire line.



case 3 - The two lines are parallel. No intersection therefore no solution.



We have already studied systems like case 1 in section 1.4. Let's look at examples of the other two cases.

case 2: We have the system of equations

$$y - 2x = 1 \quad \text{and} \quad -2y + 4x = 2$$

Find the solution.

Answer:

$y - 2x = 1$  solve for  $y$ ,  $y = 2x + 1$ . Now plug into the second equation and find

$$-2(2x + 1) + 4x = -2.$$

Simplify and get  $0 = 0$ .

This is true, so I suspect I really only have one line. Write in slope-intercept form to check - both give  $y = 2x + 1$  and so they are the same line.

How do we express the solution? We use the format  $(x, y)$  only instead of this being coordinates for a single point, it will tell me the entire line is the solution. We have one parameter in this answer and traditionally we want to use  $y$  as our parameter. So I need to find what  $x$  is in terms of  $y$ ,

$x = y/2 - 1/2$ . Put this into the solution form,

$(y/2 - 1/2, y)$  is the answer.

case 3: We have the system of equations

$$3x - 2y = 4 \quad \text{and} \quad -6x + 4y = 7$$

Find the solution.

Answer: Take the first equation and solve it for  $x$ ,

$$3x = 2y + 4 \text{ now divide by 3, } x = (2/3)y + (4/3).$$

Now plug back into the second equation,

$$-6((2/3)y + (4/3)) + 4y = 7. \text{ Simplify and find } 0 = 15.$$

This is FALSE. That means there isn't a place that both equations can be true at the same time. To check you can write both in slope-intercept form and see they have the same slope but different intercepts, so they must be parallel lines. The first equation gives  $y = (3/2)x - 2$  and the second gives  $y = (3/2)x + (7/4)$ .

## WORD PROBLEMS

It is ESSENTIAL that you are able to set up (and solve) a word problem. The most important step is the first one - DEFINE YOUR VARIABLES!! The variables are the things that you don't know. The problem will generally ask how much or how many of something. What you choose as your variables must answer that question.

example: Jane invests \$10,000 in three ways. With one part she buys mutual funds with a return of 6.5% per year. The second part (which is twice as large as the first part) is used to buy government bonds that pay 6% per year. The rest is put into a savings account paying 5% per year. In the first year her average return was 6.05%.. How much did she invest in each way?

answer: start by defining your variables,

$x$  = amount of money in mutual funds.

$y$  = amount of money in government bonds.

$z$  = amount of money in savings account.

Now find the equations,

$$x + y + z = 10,000 \text{ (total money available)}$$

$$2x = y \text{ (ratio - check with \$1 in mutual funds, } x = 1 \text{ plug this in and } 2(1) = 2 \text{ in bonds which would work)}$$

$$.065x + .06y + .05z = .0605 * 10,000 = 605 \text{ (earnings)}$$

## GAUSS-JORDAN

Start by writing our system of equations as an augmented matrix:

$$\begin{array}{l} 2x + y - z = 2 \\ x + 3y + 2z = 1 \\ x + y + z = 2 \end{array} \leftrightarrow \left[ \begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 1 & 3 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{array} \right]$$

We have the  $x$  coefficients in the first column,  $y$  coefficients in the second column,  $z$  coefficients in the third column. The equals sign is replaced with a vertical line and the last column has the constants.

We will want to change our "messy" system (matrix) to one that looks "nice", like

$$\begin{array}{l} x = a \\ y = b \\ z = c \end{array} \leftrightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array} \right]$$

We have three rules that we must follow to change the system (matrix) from messy to nice:

1. Any two equations (rows) may be interchanged.
2. An equation (row) may be multiplied by a non-zero constant.
3. A multiple of one equation (row) may be added to another equation (row).

Gauss-Jordan is simply an orderly method to use these rules to improve our matrix. We can use our calculator to help with the arithmetic. The program ROWOPS will do these operations on a matrix. The steps of G-J are:

1. Make the upper left element of the augmented matrix a 1. You can do this by swapping rows or by multiplication.
2. Use the 1 in the upper left position to make the rest of the leftmost column 0's. This is eliminating  $x$  from all but the first equation.
3. Make the second column, second row element a 1 using multiplication or row swapping (but don't move the first row!). This gives  $y$  a coefficient of 1 in the second equation.
4. Use the 1 in the second row, second column to make the other elements in that column be 0. This will eliminate  $y$  from all the other equations (rows) and keeping  $y$  only in the second equation (row).
5. Continue moving down and to the right making the next diagonal element a 1 and then the rest of the column containing that 1 zeros.
6. G-J is over when the matrix is in Row Reduced Form (RREF).

The rules for RREF are:

1. Each row consisting entirely of zeros lies below any row having non-zero entries.
2. The first non-zero entry in any row is a 1 (called a leading 1)
3. In any two successive (non-zero) rows the leading 1 in the lower row lies to the right of the leading 1 in the upper row.
4. If a column contains a leading 1, the rest of the column is 0.

So let us do our example using the ROWOPS program.

(calculator work)

After G-J is done (matrix is in RREF) we could have a matrix like,

$$\left[ \begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

First check it is in RREF form. Yes. So we must now write out our equations:

$$\begin{array}{rcl} x - z & = & 2 \\ y + 2z & = & -3 \\ 0 & = & 0 \end{array}$$

Our last statement is true, so we can ignore it. The other equations tell me the answer is parametric. We don't have enough to get a single number as an answer, so we can write the values of  $x$  and  $y$  in terms of  $z$ :

$$x = z + 2 \text{ and } y = -2z - 3. \text{ put this in } (x, y, z) \text{ form: } (z + 2, -2z - 3, z)$$

Another possibility is that when you write out the equations you get something like  $0=1$  or some other FALSE statement. Then the answer is easy - NO SOLUTION.

The RREF function on the TI-83 or RREF program on the TI-82 will automate this. You do still need to know HOW to do G-J and what RREF means. Also, the function will not work if there are more rows than columns in the augmented matrix.

## MATRICES

Matrices are a compact way of showing a large amount of data. You must be able to put data into a matrix and then add, subtract and multiply matrices. It is important to label the rows and columns of your matrix if it is a word problem. The size of a matrix is given as *number of rows*  $\times$  *number of columns*. If we are given the matrix

$$A = \begin{bmatrix} 1 & -3 & 0 \\ -2 & 6 & 4 \end{bmatrix}$$

We would say that  $A$  is a  $2 \times 3$  matrix as it has 2 rows and 3 columns.

The elements of a matrix are described in terms of its row number and column number (in that order). Element  $a_{ij}$  is in row  $i$  and column  $j$ . So element  $a_{12} = -3$ , the element in the first row and second column of matrix  $A$ .

### DATA MATRICES

A chain of shoe stores has two stores in the area, Store A and Store B. Each carries children's, women's and men's shoes. In one week Store A sold 150 pairs of children's shoes, 90 pairs of women's shoes and 40 pairs of men's shoes. Store B sold 50 pairs of children's shoes, 120 pairs of women's shoes and 60 pairs of men's shoes. Show this data in a  $2 \times 3$  matrix.

$$\begin{array}{l} \text{store A} \\ \text{store B} \end{array} \begin{array}{ccc} \text{children} & \text{women} & \text{men} \\ \left[ \begin{array}{ccc} 150 & 90 & 40 \\ 50 & 120 & 60 \end{array} \right] \end{array}$$

## MATRIX ALGEBRA

Two matrices are equal if each corresponding pair of elements in the matrices are equal.

Two matrices are added by adding the corresponding pairs of elements.

A matrix is multiplied by a constant by multiplying each element of the matrix by the constant.

example: Find the value of  $w, x, y$  and  $z$  given the following matrices:

$$2 \begin{bmatrix} x & 1 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & y \\ w & 3 \end{bmatrix} = \begin{bmatrix} 7 & -2 \\ 5 & z \end{bmatrix}$$

Start by multiplying the first matrix by 2 and then add the two matrices on the left side,

$$\begin{bmatrix} 2x & 2 \\ 4 & 0 \end{bmatrix} + \begin{bmatrix} 1 & y \\ w & 3 \end{bmatrix} = \begin{bmatrix} 2x + 1 & 2 + y \\ 4 + w & 0 + 3 \end{bmatrix} = \begin{bmatrix} 7 & -2 \\ 5 & z \end{bmatrix}$$

Now use the property of matrix equality to match the corresponding elements in the matrices:

$$2x + 1 = 7 \text{ solve and } x = 3$$

$$2 + y = -2 \text{ solve and } y = -4$$

$$4 + w = 5 \text{ solve and } w = 1$$

$$3 = z$$

## MATRIX MULTIPLICATION

If  $A$  is an  $m \times n$  matrix and  $B$  is an  $n \times p$  matrix, then the product matrix,  $AB$  is  $m \times p$ . Notice that the number of columns in the left matrix is equal to the number of rows in the right matrix.

Matrix multiplication is usually not commutative,  $AB \neq BA$ . BE CAREFUL!

Matrix multiplication involves multiplying elements and adding up the product. The algorithm is to move across the rows of the left matrix and down the columns of the right matrix. Generally this is done by your calculator, but you must know if you are multiplying the right matrices the right way.

Example: Find the product

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (1 \cdot a + 2 \cdot c) & (1 \cdot b + 2 \cdot d) \\ (3 \cdot a + 4 \cdot c) & (3 \cdot c + 4 \cdot d) \end{bmatrix}$$

Example: A contractor builds three kinds of houses, models A, B and C with a choice of two styles, Spanish or Modern. There are four different materials used on the exterior of these houses, concrete (in cubic yds), lumber (1000's of board feet), bricks (in 1000's) and shingles (in 100's of square ft). The data is summerized in the matrices below:

$$P = \begin{matrix} & \begin{matrix} \textit{Spanish} & \textit{Modern} \end{matrix} \\ \begin{matrix} \textit{Model A} \\ \textit{Model B} \\ \textit{Model C} \end{matrix} & \begin{bmatrix} 0 & 30 \\ 10 & 20 \\ 20 & 20 \end{bmatrix} \end{matrix}$$

$$Q = \begin{matrix} & \begin{matrix} \textit{concrete} & \textit{lumber} & \textit{brick} & \textit{shingles} \end{matrix} \\ \begin{matrix} \textit{Spanish} \\ \textit{Modern} \end{matrix} & \begin{bmatrix} 10 & 2 & 0 & 2 \\ 50 & 1 & 20 & 2 \end{bmatrix} \end{matrix}$$

$$R = \begin{matrix} & \begin{matrix} \textit{cost per unit} \end{matrix} \\ \begin{matrix} \textit{concrete} \\ \textit{lumber} \\ \textit{brick} \\ \textit{shingles} \end{matrix} & \begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix} \end{matrix}$$

How much of each material is needed for each model? We will want to find how much concrete is used in model A, so we will have to add up how much concrete is used in the Spanish model A and how much in the Modern model A. These kinds of sums will come from multiplying  $P$  and  $Q$ :

$$PQ = \begin{bmatrix} 0 & 30 \\ 10 & 20 \\ 20 & 20 \end{bmatrix} \times \begin{bmatrix} 10 & 2 & 0 & 2 \\ 50 & 1 & 20 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \cdot 10 + 30 \cdot 50 & \dots \\ \vdots & \ddots \end{bmatrix}$$

We get the entire answer on the calculator...

$$PQ = \begin{array}{l} \text{model A} \\ \text{model B} \\ \text{model C} \end{array} \begin{array}{cccc} \text{concrete} & \text{lumber} & \text{brick} & \text{shingles} \\ \left[ \begin{array}{cccc} 1500 & 30 & 600 & 60 \\ 1100 & 40 & 400 & 60 \\ 1200 & 60 & 400 & 80 \end{array} \right] \end{array}$$

To find the cost of the exterior materials for each model we would multiply  $PQ$  by  $R$ :

$$\begin{bmatrix} 1500 & 30 & 600 & 60 \\ 1100 & 40 & 400 & 60 \\ 1200 & 60 & 400 & 80 \end{bmatrix} \times \begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix}$$

$$= \begin{bmatrix} 20 \cdot 1500 + 30 \cdot 180 + 600 \cdot 60 + 60 \cdot 25 \\ \vdots \end{bmatrix}$$

$$= \begin{array}{l} \text{model A} \\ \text{model B} \\ \text{model C} \end{array} \begin{array}{c} \text{cost} \\ \left[ \begin{array}{c} 72,900 \\ 54,700 \\ 60,800 \end{array} \right] \end{array}$$

One special matrix is the identity matrix,  $I$ . It is  $n \times n$  and it has 1's along the diagonal (top left to lower right) and has the property that when you multiply a matrix  $A$  by  $I$  you get your original matrix  $A$  back (assuming  $A$  has the right dimensions to be multiplied). A  $4 \times 4$  identity matrix would look like

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A system of linear equations can be written as a matrix equation  $AX = B$ :

$$\begin{array}{rclcl} 2x & + & 4y & + & 6z & = & 4 \\ -x & - & 4y & - & 3z & = & 8 \\ & & y & - & z & = & -4 \end{array}$$

as a matrix equation this looks like,

$$\begin{bmatrix} 2 & 4 & 6 \\ -1 & 4 & -3 \\ 0 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ -4 \end{bmatrix}$$

## MATRIX INVERSES

The inverse of matrix  $A$  is  $A^{-1}$ . Matrix  $A$  must be square to have an inverse. If the inverse of  $A$  does not exist, matrix  $A$  is singular. The matrix inverse has the property

$$A \cdot A^{-1} = I = A^{-1} \cdot A$$

We will find the inverse of  $A$  on the calculator:

Find the inverse of

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \\ -1 & 2 & 0 \end{bmatrix}$$

Start with entering  $A$  into the calculator (say as [A]). Then back to the home screen and do matrix name [A] and then the button  $x^{-1}$  and enter. We find

$$A^{-1} = \begin{bmatrix} -4/3 & 2/3 & -1/3 \\ -2/3 & 1/3 & 1/3 \\ 7/6 & -1/3 & 1/6 \end{bmatrix}$$

Matrix inverses are used to solve matrix equations. The linear system must be written as  $AX = B$ :

$AX = B$  left multiply both sides by  $A^{-1}$

$A^{-1}AX = A^{-1}B$  use property of the inverse

$IX = A^{-1}B$  use property of the identity matrix

$X = A^{-1}B$  system is solved.

Do our example from the last section,

$$\begin{bmatrix} 2 & 4 & 6 \\ -1 & 4 & -3 \\ 0 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ -4 \end{bmatrix}$$

Now we solve to get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ -1 & 4 & -3 \\ 0 & 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 8 \\ -4 \end{bmatrix} = \begin{bmatrix} -55/3 \\ 5/3 \\ 17/3 \end{bmatrix}$$

Solve other matrix equations,

Solve  $AB + AC = C$  for  $A$ :

$$A(B + C) = C$$

$$A(B + C) \cdot (B + C)^{-1} = C(B + C)^{-1}$$

$$A \cdot I = C(B + C)^{-1}$$

$$A = C(B + C)^{-1}$$