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A Note on the Elliptic Sine-Gordon Equation

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Abstract. The elliptic sine-Gordon equation originates from the static case of the hyperbolic sine-Gordon equation modeling the Josephson junction in superconductivity. However, the elliptic sine-Gordon boundary value problem as studied in the mathematical literature actually has an opposite sign in front of the sine nonlinearity; it models not the "usual" Josephson junction but rather the Josephson π-junction, which is of contemporary interest to physicists. We first furnish this physical backdrop that has motivated our study here. Then we aim to establish the existence of nonconstant solutions of the semilinear elliptic sine-Gordon equation subject to homogeneous Neumann and Dirichlet boundary conditions by using critical point theory. Positive numerical solutions of the Dirichlet case, which are global minima of the variational problem, are computed on a dumbbell-shaped 2D domain for visualization.

1. Origin of the model

The hyperbolic sine-Gordon equation

$$\phi_{xx} - \phi_{tt} = \sin\phi \tag{1.1}$$

describes the dynamics of many condensed matter systems. Examples include: a 1-D ferromagnet with planar anisotropy in the presence of a magnetic field perpendicular to the chain direction [**Mi**], spin dynamics of the A-phase of superfluid ^{3}He [**MK**], and a Josephson transmission line [**Sc**], etc., besides such classical examples as: a chain of coupled pendula [**Dr**], and a classical model on 1-D dislocation [**La**].

The sine-Gordon equation may be derived from the Lagrangian:

$$\mathcal{L} = \int dx \left(\frac{1}{2}\phi_t^2 - \frac{1}{2}\phi_x^2 + \cos\phi - 1 \right), \tag{1.2}$$

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or from the Hamiltonian:

$$\mathcal{H} = \int dx \left(\frac{1}{2}\phi_t^2 + \frac{1}{2}\phi_x^2 + 1 - \cos\phi \right). \tag{1.3}$$

In the case of the Josephson transmission line, the variable ϕ above describes the *relative phase*, $\phi = \phi_I - \phi_{II}$, between the superconducting metals I and II, which causes the Josephson tunneling current to flow across a very thin insulating barrier. If the Josephson transmission line is wide as well as long, the hyperbolic sine-Gordon equation must be generalized to two space dimensions and one in time:

$$\phi_{xx} + \phi_{yy} - \phi_{tt} = \sin\phi\,. \tag{1.4}$$

A somewhat detailed derivation of (1.4) may be found in [**RS**](p. 30); see (1.6)–(1.10) below. The static version of this equation reads

$$\phi_{xx} + \phi_{yy} = \sin\phi; \tag{1.5}$$

cf.[**RS**] (p. 31, (1.88)).

Now, consider however, a Josephson π-junction, which is formed by either inserting a ferromagnetic layer of a sufficient thickness inside the tunneling barrier perpendicular to the current direction [**Ku**], or using d-wave high-T$_\mathrm{c}$ superconductors to form the two electrodes of the junction in an appropriate arrangement [**SR**]. Thus, the basic equations modeling the Josephson π-junction (based on the very recent papers [**Ku**] by T. Kontos *et al.* and Y. Blum *et al.*) now become

$$\frac{\partial\phi}{\partial t} = \frac{2eV}{\hbar}, \tag{1.6}$$

$$\frac{\partial\phi}{\partial x} = \left(\frac{2ed}{\hbar c}\right) H_2, \tag{1.7}$$

$$\frac{\partial\phi}{\partial y} = \left(-\frac{2ed}{\hbar c}\right) H_1, \tag{1.8}$$

$$j_3 = -j_0(x,y)\sin\phi, \tag{1.9}$$

where

$e =$ electron charge; $c =$ the speed of light in vacuo;

$d = \lambda_1 + \lambda_2 + b$, with

λ_1, λ_2 equal to, respectively, the London penetration depths for metals I and II;

$j_3(x,y) =$ the Josephson current per unit area across the barrier, with $j_0(x,y)$ depending on the properties of the barrier;

H_1, H_2 are, respectively, the x and y components of the magnetic field;

$\hbar =$ Planck's constant; and

$V(x,y,t) =$ the potential difference across the barrier.

See Fig. 1.

Substituting (1.6)–(1.9) into the Maxwell equation [**RS**] (p. 30, (1.84))

$$\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} - \frac{4\pi}{4}\cdot C\cdot\frac{\partial V}{\partial t} = \frac{4\pi}{c} j_3 \tag{1.10}$$

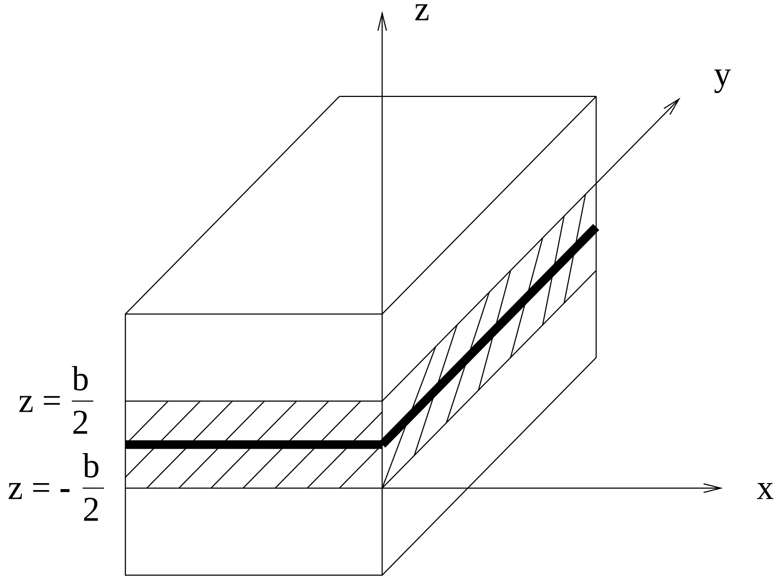

FIGURE 1. A Josephson π-junction. For $z > b/2$, there is superconducting metal I and for $z < -b/2$, superconducting metal II. The insulating barrier lying between $-b/2 < z < b/2$ (the light shaded area) is inserted with a thin layer (the dark area) of ferromagnetic material, causing j_3 in (1.10) to have a negative sign. (This is the namesake for the π-junction as it means the change of phase by π, yielding a negative sign for j_3.)

yields the barrier equation

$$\Delta\phi - \frac{1}{c_0^2}\frac{\partial^2\phi}{\partial t^2} = -\lambda_J^{-2}\sin\phi, \tag{1.11}$$

where $c_0 = c(4\pi CD)^{-1/2}$, $\lambda_J = c[\hbar/(8\pi e d j_0)]^{1/2}$, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, which, after normalization, gives

$$\phi_{xx} + \phi_{yy} - \phi_{tt} = -\sin\phi,$$

with static version

$$\phi_{xx} + \phi_{yy} = -\sin\phi. \tag{1.12}$$

This is the elliptic sine-Gordon equation we intend to investigate (rather than (1.5)) in this paper.

The boundary condition is the *natural* boundary condition

$$\frac{\partial\phi}{\partial n} = 0,$$

where n is the unit outward normal vector on the boundary. In order to secure the Dirichlet boundary condition $\phi = 0$ or the Robin condition $\phi + \alpha\partial\phi/\partial n = 0$, it is apparent that some devices or controls must be imposed thereupon. We have not yet found much discussion about how to carry out the design of such devices or controls in the literature, however.

In this paper, we present some theoretical and numerical results for the elliptic sine-Gordon boundary value problem. We will discuss both the homogeneous Dirichlet and Neumann boundary conditions, with the understanding that the discussion so far for the homogeneous Dirichlet condition is primarily of academic interest. Even though the existence of multiple solutions of a general class of nonlinear eigenvalue problems has already been discussed in [**Ra**] (in particular, Theorem 9.6), however, we must profess that for the elliptic sine-Gordon equation, the understanding of qualitative properties of the nontrivial solutions rather than the existence of multiple nontrivial solutions seems to be limited, at least as far as the authors are concerned. Furthermore, this paper should be viewed more as a research in progress rather than a major milestone on this subject.

2. The elliptic sine-Gordon equation (*I*): Neumann boundary condition

Let Ω be a bounded open domain in $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$. Let $H^s(\Omega)$ denote the usual Sobolev space of order $s \in \mathbb{R}$. Consider the boundary value problem (BVP)

$$\begin{cases} -\Delta w(x) = \lambda \sin w(x), & x = (x_1, \dots, x_N) \in \Omega, \quad \lambda > 0, \\ \dfrac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

The variational functional corresponding to (2.1) is

$$J(v) = \int_\Omega \left[\frac{1}{2}|\nabla v|^2 - \lambda(1 - \cos v)\right] dx, \qquad v \in H^1(\Omega). \tag{2.2}$$

Sometimes, for clarity we will write J as J_λ to signify its dependence on λ.

REMARK 2.1. By making the change of variable

$$w(x) = \pi - v(x), \qquad x \in \Omega, \tag{2.3}$$

in (2.1), we obtain

$$\begin{cases} -\Delta v(x) = -\lambda \sin v(x), & x \in \Omega, \\ \dfrac{\partial v}{\partial n} = 0 & \text{on } \Omega. \end{cases} \tag{2.4}$$

Thus, we see that as far as the Neumann BVP is concerned, mathematically speaking, the Josephson π-function and the usual Josephson junction are *equivalent.*

However, for the homogeneous Dirichlet BVP, the change of variable (2.3) is inadmissible, and thus the equations for the Josephson π-junction and the usual Josephson junction are *not equivalent.* □

Using the calculus of variations, one can verify that if $w_0 \in H^1(\Omega)$ is a critical point of J, i.e., $J'(w_0) = 0$, then $w_0 \in H^2(\Omega)$ and w_0 is a classical solution of (2.1), and vice versa that if w_0 satisfies (2.1), then w_0 is a critical point of J satisfying $J'(w_0) = 0$.

LEMMA 2.2. *$J(v)$ has a local minimum at $v = \pi$ and, consequently, at every $(2n-1)\pi$ for every $n = 0, \pm 1, \pm 2, \dots$.*

PROOF. Let $\varepsilon > 0$. For $v_0 \in H^1(\Omega)$, define

$$B(v_0, \varepsilon) = \{v \in H^1(\Omega) \mid \|v - v_0\|_{H^1(\Omega)} < \varepsilon\}.$$

For $v \in \partial B(0,\varepsilon)$, we have

$$\begin{aligned} J(\pi+v)-J(\pi) &= \left\{ \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \lambda \int_\Omega [1-\cos(\pi+v)]dx \right\} + 2\lambda \int_\Omega \, dx \\ &= \frac{1}{2}\int_\Omega |\nabla v|^2 dx + \lambda \int_\Omega (1-\cos v) dx \\ &= \frac{1}{2}\int_\Omega |\nabla v|^2 dx + \lambda \int_\Omega \left[\frac{v^2}{2!} - \frac{v^4}{4!} + \frac{v^6}{6!} - \cdots \right] dx \\ &\geq \frac{1}{2}\int_\Omega |\nabla v|^2 dx + \frac{\lambda}{4}\int_\Omega v^2 \, dx, \end{aligned}$$

if $\varepsilon < \varepsilon_0$ for some small $\varepsilon_0 > 0$. Thus

$$J(\pi+v)-J(\pi) \geq \min\left\{\frac{1}{2}, \frac{\lambda}{4}\right\} \|v\|^2_{H^1(\Omega)}.$$

Hence

$$J(v)-J(\pi) \geq \delta_0, \text{ for some } \delta_0 > 0 \text{ for all } v \in \partial B(\pi,\varepsilon). \qquad \square$$

Let $\{\lambda_j \mid 0 = \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, j = 1,2,\ldots\}$ be the set of eigenvalues of the Neumann boundary value problem

$$\left\{ \begin{array}{ll} -\Delta\phi_j = \lambda_j \phi_j & \text{on } \Omega, \\ \dfrac{\partial \phi_j}{\partial n} = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{2.5}$$

THEOREM 2.3. *If $\lambda > \lambda_2$, then $J_\lambda(v)$ has at least two nonconstant critical points w of mountain-pass type in $H^1(\Omega)$. Consequently, $w(x) + 2k\pi$ are also solutions of* (2.1) *for any integer k.*

PROOF. Since $J \in C^1(H^1(\Omega),\mathbb{R})$ and it is straightforward to verify that J satisfies the Palais–Smale (PS) condition, by Lemma 2.2 and the Mountain Pass Lemma [**AR**], we see that $J(v)$ has at least a critical point $w_0 \in H^1(\Omega)$ such that

$$J(w_0) = \inf_{\gamma\in\Gamma} \sup_{v \in \gamma([0,1])} J(v) > \delta_0 > 0, \tag{2.6}$$

where $\Gamma = \{\gamma \in C(\mathbb{R}, H^1(\Omega)) \mid \gamma(0) = \pi, \gamma(1) = -\pi\}$.

Next, we prove that if $\lambda > \lambda_2$, then $w_0 \not\equiv 0$. For ϕ_2 satisfying (2.5), we have $\int_\Omega \phi_2 \, dx = 0$. If $\lambda > \lambda_2$, then

$$\begin{aligned} J(t\phi_2) &= \frac{t^2}{2}\int_\Omega |\nabla \phi_2|^2 dx - \lambda \int_\Omega (1-\cos(t\phi_2))dx \\ &= \frac{\lambda_2 t^2}{2}\int_\Omega \phi_2^2 \, dx - \lambda \int_\Omega \left[\frac{t^2\phi_2^2}{2!} - \frac{t^4\phi_2^4}{4!} + \frac{t^6\phi_2^6}{6!} - \cdots \right] dx \\ &= t^2 \left\{ \frac{(\lambda_2 - \lambda)}{2}\int_\Omega \phi_2^2 \, dx + t^2\lambda \int_\Omega \left[\frac{\phi_2^4}{4!} - \frac{t^2\phi_2^6}{6!} + \cdots \right] dx \right\}. \end{aligned}$$

Thus

$$< J''(0)\phi_2,\ \phi_2 >_{H^1} = (\lambda_2 - \lambda) \int_\Omega \phi_2^2 dx < 0. \tag{2.7}$$

Note that $\phi_1 = 1$. From the definition of J in (2.2), one can verify easily that

$$< J''(0)\phi_1,\ \phi_1 >_{H^1} = -\lambda|\Omega| < 0. \tag{2.8}$$

Note that ϕ_1 and ϕ_2 are perpendicular in $H^1(\Omega)$. Thus the Morse index of J at $w = 0$, which is defined by the dimension of the maximal subspace of $H^1(\Omega)$ where $J''(0)$ is negative definite [**Ch**], is greater than or equal to 2. However, since w_0 is a critical point of J of Mountain-Pass type, the Morse index of J at w_0 is less than or equal to 1 [**Ch**]. Thus, $w_0 \not\equiv 0$. Therefore, $J_\lambda(v)$ has two critical points, $\pm w_0$, of the mountain pass type, if $\lambda > \lambda_2$. ☐

By using a different approach based upon the Mountain Pass Lemma in order intervals [**LW**], C. Li has studied a class of nonlinear elliptic equations similar to the elliptic sine-Gordon equation in a recent preprint [**Li**]. By applying Theorem 1.1 in [**Li**], (2.1) admits two nonconstant solutions between $-\pi$ and π if $\lambda > \lambda_2$. However, it is unclear whether those two solutions are of mountain-pass type as what we claimed in Theorem 2.3. Our proof here is somewhat more elementary and easier to understand.

The following theorem gives us some additional information about the bounds of the solutions of (2.1).

THEOREM 2.4. *The BVP* (2.1) *does not admit any nonconstant solution w such that* $0 < w(x) < \pi$ *on Ω, for any $\lambda > 0$. Consequently, for any $\lambda > 0$,* (2.1) *does not admit any nonconstant solution w such that*

$$2n\pi < w(x) < (2n+1)\pi,$$

for any integer n.

PROOF. Assume that (2.1) admits a nonconstant solution $w(x)$ such that $0 < w(x) < \pi$, then one can easily check $v(x) = \pi - w(x)$ is a solution of (2.4). Multiplying the first equation of (2.4) by v and integrate over Ω, we have

$$-\int_\Omega |\nabla v|^2 dx = \lambda \int_\Omega v \sin v \ dx \geq 0.$$

Thus

$$\int_\Omega |\nabla v|^2 dx = 0, \quad \text{and} \quad \int_\Omega v \sin v \ dx = 0,$$

implying $v \equiv 0$ or $v \equiv \pi$, a contradiction. ☐

3. The elliptic sine-Gordon equation (*II*): Dirichlet boundary condition

For the Dirichlet BVP

$$\begin{cases} \Delta w + \lambda \sin w = 0 & \text{on } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

the variational functional J remains the same as in (2.2), but the underlying Hilbert space becomes $H_0^1(\Omega) = \{w \in H^1(\Omega) \mid w = 0 \text{ on } \partial\Omega\}$. We let $\{\tilde\lambda_j \mid 0 < \tilde\lambda_1 < \tilde\lambda_2 <$

$\cdots < \tilde{\lambda}_n < \cdots,\ j = 1, 2, 3, \ldots\}$ denote the set of eigenvalues of the homogeneous Dirichlet BVP

$$\begin{cases} -\Delta\tilde{\phi}_n = \tilde{\lambda}_n\tilde{\phi}_n, & \text{on } \Omega, \\ \tilde{\phi}_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

It is straightforward to verify that $J(0) = 0$, $J \in C(H_0^1(\Omega), \mathbb{R})$ and that solutions w of (3.1) correspond exactly to critical points w of J satisfying $J'(w) = 0$.

Proposition 3.1. *Let* λ *satisfy* $0 < \lambda < \tilde{\lambda}_1$. *Then* $w \equiv 0$ *is the only critical point of* J_λ.

Proof. Assume $w \in H_0^1(\Omega)$ is a critical point of J and $w \neq 0$. Then w is a weak solution of (3.1). By multiplying (3.1) by w and integrating over Ω, one has

$$0 < \int_\Omega |\nabla w|^2 dx = \lambda \int_\Omega w \sin w dx \leq \lambda \int_\Omega w^2 dx.$$

By the Poincaré inequality, one then has

$$0 < \int_\Omega |\nabla w|^2 dx \leq \frac{\lambda}{\tilde{\lambda}_1} \int_\Omega |\nabla w|^2 dx.$$

Thus $\lambda \geq \tilde{\lambda}_1$ which contradicts to the assumption $0 < \lambda < \tilde{\lambda}_1$. Therefore $w = 0$ is the only critical point of J_λ if $0 < \lambda < \tilde{\lambda}_1$. $\square$

For any $v \in H_0^1(\Omega)$, we have

$$J(v) = \int_\Omega \left[\frac{1}{2}|\nabla v|^2 - \lambda(1 - \cos v)\right] dx \geq \frac{1}{2}\int_\Omega |\nabla v|^2 dx - 2\lambda|\Omega|, \tag{3.3}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. Also, for any $v \in H_0^1(\Omega)$, $v \neq 0$, $J(tv)$ is continuous with respect to $t \in \mathbb{R}$ and

$$\lim_{t\to\pm\infty} J(tv) = \infty.$$

So $\min\limits_{t\in\mathbb{R}} J(tv)$ exists and there is a $t^* \in \mathbb{R}$ such that

$$J(t^*v) = \min_{t\in\mathbb{R}} J(tv).$$

We further deduce that

$$m \equiv \inf_{v\in H_0^1(\Omega)} \min_{t\in\mathbb{R}} J(tv) = \min_{v\in H_0^1(\Omega)} \min_{t\in\mathbb{R}} J(tv) \geq -2\lambda|\Omega|.$$

Let $\{v_n^* \in H_0^1(\Omega) \mid n = 1, 2, 3, \ldots\}$ be such that

$$J(v_n^*) = \min_{t\in\mathbb{R}} J(tv_n^*),$$
$$\lim_{n\to\infty} J(v_n^*) = m.$$

Then, since

$$J(v_n^*) \geq \frac{1}{2}\int_\Omega |\nabla v_n^*|^2 dx - 2\lambda|\Omega|,$$

we deduce that $\{v_n^* \mid n = 1, 2, 3, \ldots\}$ is bounded in $H_0^1(\Omega)$. Thus $\{v_n^*\}$ has a weakly convergent subsequence, still denoted the same, such that

$$\lim_{n\to\infty} v_n^* = v_0^* \quad \text{weakly in} \quad H_0^1(\Omega). \tag{3.4}$$

Since J is continuously Fréchet differentiable, we have

$$m \le J(v_0^*) \le \lim_{n\to\infty} J(v_n^*) = m = \min_{v\in H_0^1(\Omega)} J(v). \tag{3.5}$$

We summarize the above in the following.

Lemma 3.2. *Let $\lambda > 0$. Then $J_\lambda(\cdot)$ has a global minimizer v_0^* in $H_0^1(\Omega)$ satisfying (3.1). If $v_0^* \not\equiv 0$, then $-v_0^*$ is also a global minimizer.* ☐

Theorem 3.3. *If $\lambda > \tilde\lambda_1$, then the global minimizer v_0^* in Lemma 3.2 is nontrivial: $v^* \not\equiv 0$. Therefore, $-v_0^*$ is also a distinct global minimizer.*

Proof. We need only to prove that there is a $v_0 \ne 0$, $v_0 \in H_0^1(\Omega)$ such that $J(v_0) < 0 = J(0)$.

Let $v_0 = t\tilde\phi_1$, cf. (3.2). Then

$$\begin{aligned} J(t\tilde\phi_1) &= \frac{t^2}{2}\int_\Omega |\nabla\tilde\phi_1|^2 dx - \lambda \int_\Omega (1-\cos(t\tilde\phi_1))dx \\ &= \frac{\tilde\lambda_1 t^2}{2}\int_\Omega |\tilde\phi_1|^2 dx - \lambda \int_\Omega (1-\cos(t\tilde\phi_1))dx. \end{aligned}$$

Using

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}\cdots\cdots, \qquad x\in\mathbb{R},$$

we have

$$J(t\tilde\phi_1) = \frac{1}{2}(\tilde\lambda_1 - \lambda)t^2 \int_\Omega |\tilde\phi_1|^2 dx + \lambda\int_\Omega \left[\frac{t^4|\tilde\phi_1|^4}{4!} - \cdots\right] dx.$$

Since $\tilde\phi_1$ is C^∞ on Ω, $\|\tilde\phi_1\|_{L^\infty(\Omega)} < +\infty$, and

$$\begin{aligned} J(t\tilde\phi_1) &= t^2\left\{-\frac{1}{2}(\lambda-\tilde\lambda_1)\int_\Omega |\tilde\phi_1|^2 dx + \lambda t^2 \int_\Omega \left[\frac{|\tilde\phi_1|^4}{4!} - \frac{t^2|\tilde\phi_1|^6}{6!} + \cdots\right] dx\right\} \\ &\equiv t^2\{T_1 + \lambda t^2 T_2\}, \end{aligned}$$

where T_1 is a negative constant, and T_2 depends on t such that for $|t|$ small, T_2 is positive.

Thus, when t is small enough, i.e., there exists $\delta_0 > 0$ such that

$$J(t\tilde\phi_1) < 0 \quad \text{if} \quad |t| < \delta_0. \tag{3.6}$$

So the proof is complete. ☐

Theorem 3.4. *Let $\lambda > \tilde\lambda_1$. If w_0 is global minimizer of J, then w_0 is of one sign.*

Proof. Note that w_0 is a classical solution of (3.1), and $w_0 \in H^2(\Omega)\cap H_0^1(\Omega)$ by regularity theory. Write

$$w_0(x) = w_0^+(x) + w_0^-(x); \quad w_0^+ = \max(w_0, 0), \quad w_0^- = \min(w_0, 0).$$

Thus $w_0^+ \in H_0^1(\Omega), w_0^- \in H_0^1(\Omega)$. It is easy to check that

$$J(w_0) = J(w_0^+) + J(w_0^-).$$

If w_0 changes signs on Ω, then

$$J(w_0^+) < J(w_0), \quad J(w_0^-) < J(w_0),$$

contradicting that w_0 is a global minimizer of J on $H_0^1(\Omega)$. Thus either $w_0 = w_0^+$ or $w = w^-$. $\square$

Theorem 3.5. *Let $\lambda > \tilde{\lambda}_1$. If $w_0(x) \geq 0$ is the global minimizer of J on $H_0^1(\Omega)$, then $0 < w_0(x) < \pi$ on Ω.*

Proof. Assume $w_0(x) \geq 0$ is the global minimizer of J on $H_0^1(\Omega)$, and $\max_{x\in\Omega} w_0(x) > \pi$. Thus, define

$$w_\pi(x) = \begin{cases} \pi, & \text{if} \quad w_0(x) \geq \pi, \\ w_0(x), & \text{if} \quad w_0(x) < \pi. \end{cases}$$

We have $w_\pi \in H_0^1(\Omega)$, and

$$\begin{aligned} J(w_0) - J(w_\pi) &= \left[\frac{1}{2}\int\limits_\Omega |\nabla w_0|^2 dx - \lambda \int\limits_\Omega (1 - \cos w_0) dx\right] \\ &\quad - \left[\frac{1}{2}\int\limits_\Omega |\nabla w_\pi|^2 dx - \lambda \int\limits_\Omega (1 - \cos w_\pi) dx\right] \\ &= \frac{1}{2}\int\limits_{\Omega^*} |\nabla w_0|^2 dx - \lambda \int\limits_{\Omega^*} (\cos w_\pi - \cos w_0) dx, \end{aligned}$$

where $\Omega^* = \{w \in \Omega \mid w_0(x) \geq \pi\}$, giving

$$\begin{aligned} J(w_0) - J(w_\pi) &= \frac{1}{2}\int\limits_{\Omega^*} |\nabla w_0|^2 dx - \lambda \int\limits_{\Omega^*} (-1 - \cos w_0) dx \\ &= \frac{1}{2}\int\limits_{\Omega^*} |\nabla w_0|^2 dx + \lambda \int\limits_{\Omega^*} (1 + \cos w_0) dx > 0, \end{aligned}$$

if $w(x)$ crosses π, i.e., $|\Omega^*| > 0$. This contradicts the fact that w_0 is a global minimizer of J on $H_0^1(\Omega)$. Hence $0 \leq w_0(x) \leq \pi$. By a result of Ambrosetti and Hess [**AH**],

$$\|w_0\|_\infty \neq \pi.$$

Thus $0 \leq w_0(x) < \pi$. By the maximum principle, we have $w_0(x) > 0$ on Ω. Therefore, $0 < w_0(x) < \pi$ on Ω. $\square$

Theorem 3.5 indicates that any global minimizer of J_λ for $\lambda > \tilde{\lambda}_1$ is naturally positive on Ω. In the following, we provide a few additional theorems on such a "positive solution."

Theorem 3.6. *Let $\lambda > \tilde{\lambda}_1$. Assume that $w_0 > 0$ is a global minimizer of J on $H_0^1(\Omega)$. Then w_0 is a maximal solution of* (3.1) *in the sense that if $w > 0$ also satisfies* (3.1) *and $\|w\|_\infty < \pi$, then $w \leq w_0$ on Ω, where $\| \ \|_\infty$ denotes the $L^\infty(\Omega)$-norm.*

Proof. From the proof of Theorem 3.5, we know that $\|w_0\|_\infty < \pi$. Rewrite (3.1) as

$$\begin{cases} (-\Delta + \lambda_0)w_0 = \lambda_0 w_0 + \lambda \sin w_0, & \text{on } \Omega \\ w_0\big|_{\partial\Omega} = 0; \quad w_0 > 0 & \text{on } \Omega, \end{cases} \tag{3.7}$$

where $\lambda_0 > \lambda$ is a given number. Define $\mathcal{A}_{\lambda_0} = -\Delta + \lambda_0$ on $H_0^1(\Omega) \cap H^2(\Omega)$, and $f(u) = \lambda_0 u + \lambda \sin u$. One can easily check that the following holds:

(1) $\sigma(\mathcal{A}_{\lambda_0}) = \{\lambda_0 + \tilde{\lambda}_n \mid n = 1, 2, 3, \ldots, \text{ cf. } (3.2)\}$, where σ denotes the spectrum;

(2) $f(u)$ is monotonically increasing, i.e., if $u < v$, $f(u) < f(v)$.

Let $\overline{w}_0 = \pi$ on Ω, then $\overline{w}_0$ is an upper solution of (3.7), while w_0 can be viewed as an lower solution of (3.7). Thus define

$$\overline{w}_{n+1} = \mathcal{A}_{\lambda_0}^{-1} f(\overline{w}_n).$$

Then $\{\overline{w}_n\}$ is a monotone sequence such that

$$w_0 \le \overline{w}_{n+1} \le \overline{w}_n \le \cdots \le \overline{w}_0 = \pi.$$

Thus by the Monotone Iteration Theorem, there is a $\overline{w} \in H_0^1(\Omega) \cap C^\infty(\Omega)$ such that

$$\lim_{n\to\infty} \overline{w}_n = \overline{w}.$$

Thus $\overline{w}$ is a solution of (3.7), and

$$w_0 \le \overline{w} \quad \text{on} \quad \Omega;$$

$\overline{w}$ is a maximal solution of (3.7) on Ω satisfying $\|\overline{w}\|_\infty < \pi$. Next, we show $w_0 \equiv \overline{w}$. Note that

$$\begin{aligned} J(w_0) &= \frac{1}{2}\int_\Omega |\nabla w_0|^2 dx - \lambda \int_\Omega (1 - \cos w_0) dx \\ &= -\frac{1}{2}\int_\Omega \Delta w_0 \cdot w_0 \; dx - \lambda \int_\Omega (1 - \cos w_0) dx \\ &= \frac{1}{2}\int_\Omega \lambda \sin w_0 \cdot w_0 \; dx - \lambda \int_\Omega (1 - \cos w_0) dx \\ &= \lambda \int_\Omega \left[\frac{1}{2} w_0 \sin w_0 - (1 - \cos w_0)\right] dx. \end{aligned}$$

Similarly,

$$J(\overline{w}) = \lambda \int_\Omega \left[\frac{1}{2}\overline{w} \sin \overline{w} - (1 - \cos \overline{w})\right] dx.$$

Note that $g(x) = \frac{1}{2} x \sin x - (1 - \cos x)$ is monotonically decreasing on $[0, \pi]$ and, thus, $w_0 \le \overline{w}$ implies

$$J(w_0) \ge J(\overline{w}).$$

But w_0 is a global minimizer of J. We thus conclude

$$w_0 \equiv \overline{w} \quad \text{on} \quad \Omega.$$

Therefore, w_0 is a maximal solution in the sense as stated in the theorem. □

Corollary 3.7. *Let $\lambda > \tilde{\lambda}_1$. Let w_0 be a maximal solution satisfying the assumptions of Theorem 3.6. Assume that $w > 0$ also satisfies* (3.1) *and $\|w\|_\infty < \pi$. Then $w \equiv w_0$ on Ω, i.e.,* (3.1) *has a unique positive solution satisfying $\|w\|_\infty < \pi$.*

Proof. From Theorem 3.6, we have $w(x) \le w_0(x)$, $x \in \Omega$. Thus

$$\begin{aligned} 0 &= \int\limits_\Omega [(-\Delta w_0)w - (-\Delta w)w_0]dx \\ &= \int\limits_\Omega [\lambda \sin w_0 \cdot w - \lambda \sin w \cdot w_0]dx \\ &= \lambda \int\limits_\Omega w w_0 \left[\frac{\sin w_0}{w_0} - \frac{\sin w}{w}\right] dx. \end{aligned}$$

Note that $w(x)w_0(x) > 0$ on Ω, and $\frac{\sin x}{x}$ is monotonically decreasing on $[0, \pi]$, thus,

$$0 = \lambda \int\limits_\Omega w w_0 \left[\frac{\sin w_0}{w_0} - \frac{\sin w}{w}\right] dx$$

implying $w w_0 \left[\frac{\sin w_0}{w_0} - \frac{\sin w}{w}\right] \equiv 0$ on Ω and hence $w = w_0$ on Ω. □

Example 3.8 (Uniqueness of the positive solution on the disk). Let $\lambda > \tilde{\lambda}_1$. Consider

$$\begin{cases} \Delta u = -\lambda \sin u & \text{on } \Omega, \\ u > 0 & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.8}$$

where $\Omega = \{(x_1, x_2) \mid x_1^2 + x_2^2 < R^2\}$ for some $R > 0$. By results in Gidas, Ni and Nirenberg [**GNN**], all solutions of (3.8) are radially symmetric. Thus, let

$$w(r) = u(r, \theta);$$

$w(r)$ satisfies

$$\begin{cases} \dfrac{d^2 w}{dr^2} + \dfrac{1}{r}\dfrac{dw}{dr} = -\lambda \sin w, & 0 \le r < R, \\ w(r) > 0, & 0 \le r < R, \\ w'(0) = 0, \quad w(R) = 0. \end{cases} \tag{3.9}$$

Multiply both sides of (3.9) by $\frac{dw}{dr}$, and integrate from 0 to r,

$$\frac{1}{2}\left(\frac{dw}{dr}\right)^2 + \int_0^r \frac{1}{r}\left(\frac{dw}{dr}\right)^2 dr = \lambda(\cos w - \cos w(0)), \qquad 0 \le r < R.$$

Since the left-hand side is always nonnegative, we must have $\cos w(0) = -1$ if $w(r)$ crosses π, in order to maintain the nonnegativity of the left-hand side. If $w(r)$ is a solution of (3.9) and $w(r)$ crosses π, then

$$\frac{1}{2}\left(\frac{dw}{dr}\right)^2 + \int_0^r \frac{1}{r}\left(\frac{dw}{dr}\right)^2 dr = \lambda(1 + \cos w).$$

If $w(r)$ crosses π at $0 < r_0 < R$, i.e. $w(r_0) = \pi$, then

$$\frac{1}{2}\left(\frac{dw}{dr}\right)^2 (r_0) + \int_0^{r_0} \frac{1}{r}\left(\frac{dw}{dr}\right)^2 dr = \lambda(1 + \cos w(r_0)) = 0.$$

Thus $\frac{dw}{dr}(r) = 0$, $0 \le r \le r_0$. Then we deduce $w(r) \equiv \pi$, $0 \le r \le r_0$. Therefore

$$0 < w(r) \le \pi \quad \text{if} \quad 0 \le r < R.$$

By [**AH**], we have

$$\|w\|_\infty \not\equiv \pi.$$

Thus, for $\lambda > \tilde{\lambda}_1$, a positive solution w satisfies

$$0 < w(r) < \pi.$$

Consequently, by Corollary (3.7), the positive solution is unique. ☐

The preceding theorems and Example 3.8 all offer strong support to the conjecture that there is a *unique* positive solution to (3.1), if $\lambda > \tilde{\lambda}_1$. However, a complete proof is not available at this moment.

Sign-changing solutions of the elliptic sine-Gordon equation are of considerable significance because they induce the so-called Josephson vortex and, thus, corresponds to physical phenomena of practical interest. So let us consider sign-changing solutions of (3.1) in the rest of this section.

Theorem 3.9. *Let $\lambda > \tilde{\lambda}_2$. Then* (3.1) *admits at least four nontrivial solutions.*

Proof. From Theorem 3.3, (3.1) admits at least two solutions w_0 and $-w_0$, where $w_0 \in H_0^1(\Omega)$ is a global minimizer of (2.2) on $H_0^1(\Omega)$, for $\lambda > \tilde{\lambda}_1$.

Since $J(v)$ satisfies the conditions in the Mountain Pass Lemma, and w_0 and $-w_0$ are two global minimizers of $J(v)$ with $J(w_0) = J(-w_0)$, we see that $J(v)$ has at least one critical point w_1 such that

$$J(w_1) = \min_{\gamma \in \Gamma} \sup_{v \in \gamma([0,1])} J(v) > J(w_0),$$

where

$$\Gamma = \{\gamma \in C([0,1], H_0^1(\Omega)) \mid \gamma(0) = w_0, \gamma(1) = -w_0\}.$$

The only thing left is to prove that $w_1 \ne 0$.

Consider $J(v)$ near 0, let $\tilde{\phi}_2$ be the eigenfunction of $-\Delta$ on Ω corresponding $\lambda = \tilde{\lambda}_2$ cf. (3.2). By repeating the same argument in the proof of Theorem 3.3, we have

$$J(t\phi_1) = -\frac{t^2}{2}(\lambda - \tilde{\lambda}_1) \int\limits_\Omega |\tilde{\phi}_1|^2 dx + \lambda t^4 \int\limits_\Omega \left[\frac{|\tilde{\phi}_1|^4}{4!} - \frac{t^2 |\tilde{\phi}_1|^6}{6!} + \cdots \right] dx,$$

and

$$J(t\phi_2) = -\frac{t^2}{2}(\lambda - \tilde{\lambda}_2) \int\limits_\Omega |\tilde{\phi}_2|^2 dx + \lambda t^4 \int\limits_\Omega \left[\frac{|\tilde{\phi}_2|^4}{4!} - \frac{t^2 |\tilde{\phi}_2|^6}{6!} + \cdots \right] dx.$$

Since $\tilde{\phi}_1$ and $\tilde{\phi}_2$ correspond to different eigenvalues, they are linearly independent in $H_0^1(\Omega)$. Thus the Morse index of $J(v)$ at $v = 0$ is greater than or equal to 2, if $\lambda > \tilde{\lambda}_2$. Since w_1 is the mountain-pass type critical point of $J(v)$, one has that the Morse index of $J(v)$ at $v = w_1$ is less than or equal to 1. Thus $w_1 \ne 0$ if $\lambda > \lambda_2$. Therefore, (3.1) admits four nontrivial solutions: $\pm w_0$ and $\pm w_1$. By applying the techniques in [**CCN**], one can show that w_1 is sign-changing. Due to the limit of length of this paper, we omit the details. ☐

In fact, upon applying a general theorem (Theorem 9.6) in Rabinowitz [**Ra**] we can easily check the four assumptions therein are satisfied by (3.1). Thus we have

THEOREM 3.10. *If $\lambda > \tilde{\lambda}_k$, then* (3.1) *admits at least k different pairs of nontrivial solutions.*

We note that even though our Theorem 3.9 appears to be only a special case of Theorem 3.10, our proof contains slightly more information about the sign-changing solutions than what the approach in [**Ra**] would provide.

4. Visualization of positive solutions of the elliptic sine-Gordon equation subject to the homogeneous Dirichlet boundary condition

In this section, we compute positive solutions by minimizing J on $H_0^1(\Omega)$ for a dumbbell-shaped domain in 2D. The algorithm (based on the method of steepest descent) for finding the global minimum of J as guaranteed by Theorems 3.3 and 3.4 proceed as follows (assuming $\lambda > \tilde{\lambda}_1$):

(1) Choose an initial guess $u_0 \in H_0^1(\Omega)$ such that

$$0 < u_0(x) \le \pi, \qquad J(u_0) < 0;$$

(2) Find a $T > 0$ such that $J(Tu_0) \ge 0$;

(3) Find a $t^* \in (0, T)$ such that

$$J(t^* u_0) = \min_{t \in [0,T]} J(tu_0);$$

(4) Let u_0 be replaced by $t^* u_0$. Compute $v_0 = \nabla J(u_0)$ by

$$\begin{cases} \Delta v = \Delta u_0 - \lambda \sin u_0, & \text{on } \Omega, \\ v = 0, & \text{on } \partial\Omega; \end{cases} \tag{4.1}$$

(5) If $\|v\| < \varepsilon$, then output u_0; otherwise, solve

$$s = \arg\min_{s>0} J(u_0 - sv),$$

replace u_0 by $u_0 - sv$, and goto Step (3).

The dumbbell-shaped domain Ω is given in Fig. 2. It is formed by a disk on the left, centered at $(-1, 0)$ with radius $1/2$, connected through a rectangular corridor with width $1/2$, to a disk on the right centered at (2,0) with radius 1. A boundary element method [**CZ**] is used to compute the numerical solution of the linear elliptic BVP (4.1) in Step (4). Five values of λ are used:

$$\lambda = 20, \quad 25, \quad 50, \quad 100, \quad \textit{and} \quad 500.$$

Their graphics are given, in sequential order, in Figs. 3−7, along with the values of J and the maximum value of the function. We see that such solutions' maxima converge to a value close to $\pi = 3.1416$.

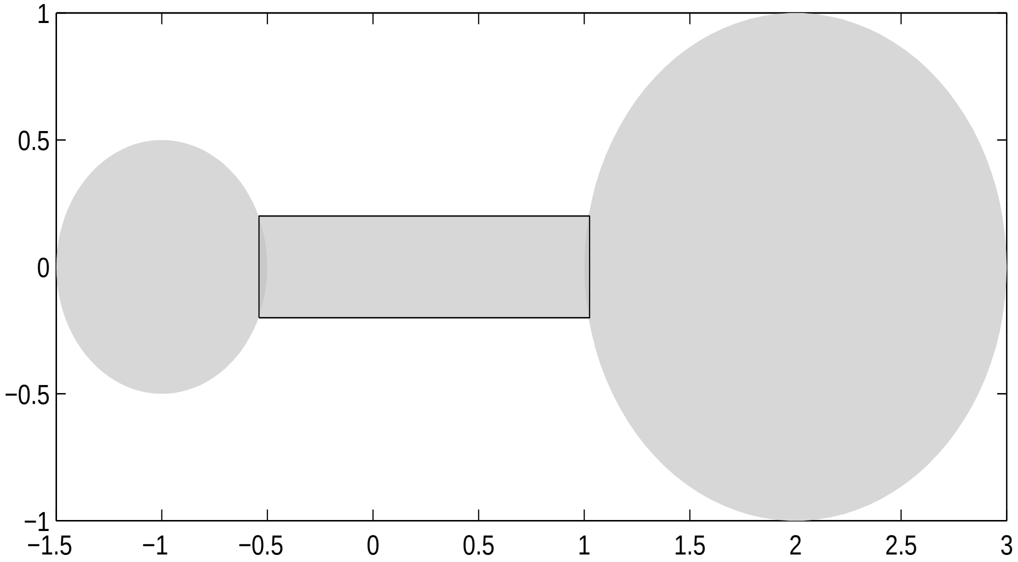

FIGURE 2. The Dumbbell-shaped domain Ω

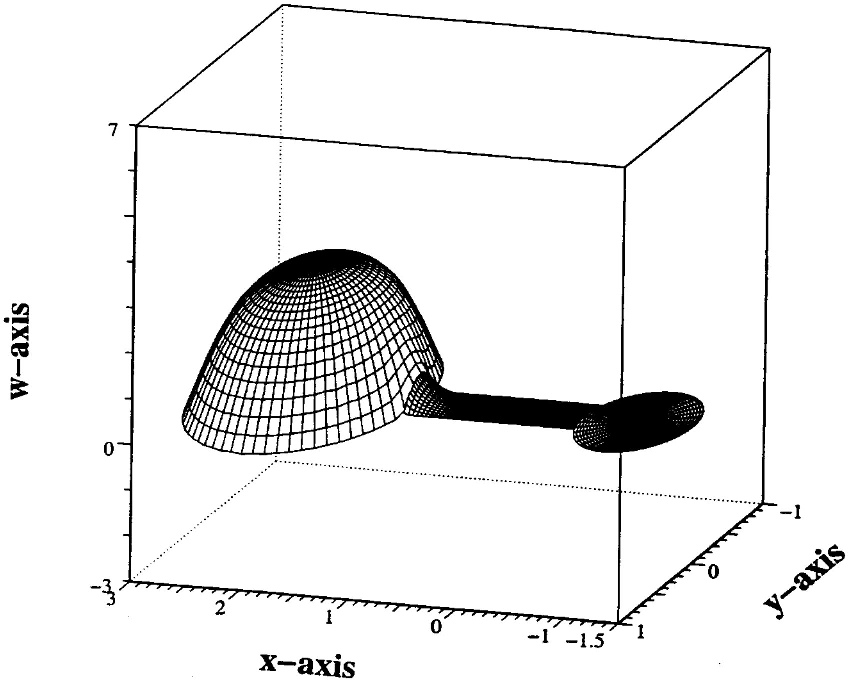

FIGURE 3. Profile of the global minimizer w for J_λ on $H_0^1(\Omega)$, $\lambda = 20$, $J_\lambda(w) = -33.47$, $\max w = 2.883$.

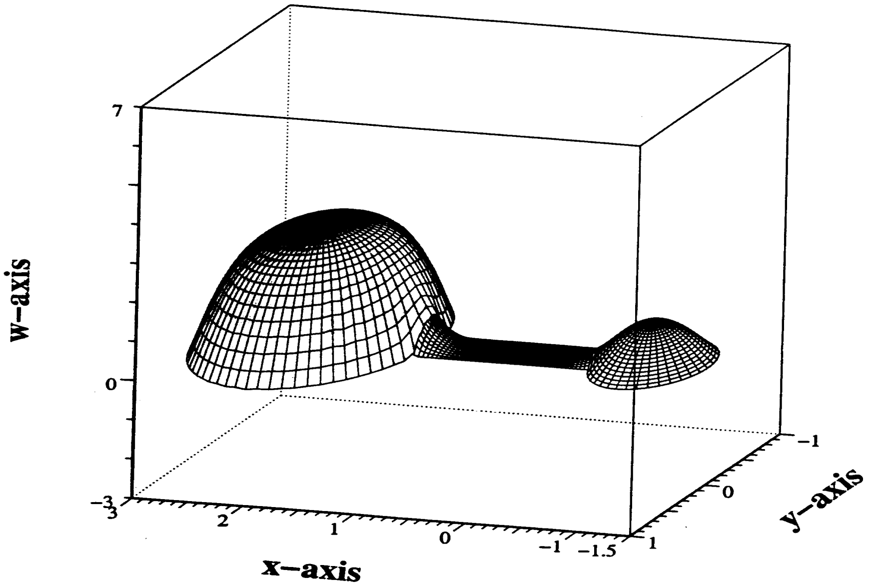

FIGURE 4. Global minimizer w, $\lambda = 25$, $J_\lambda(w) = -51.73$, $\max w = 2.984$.

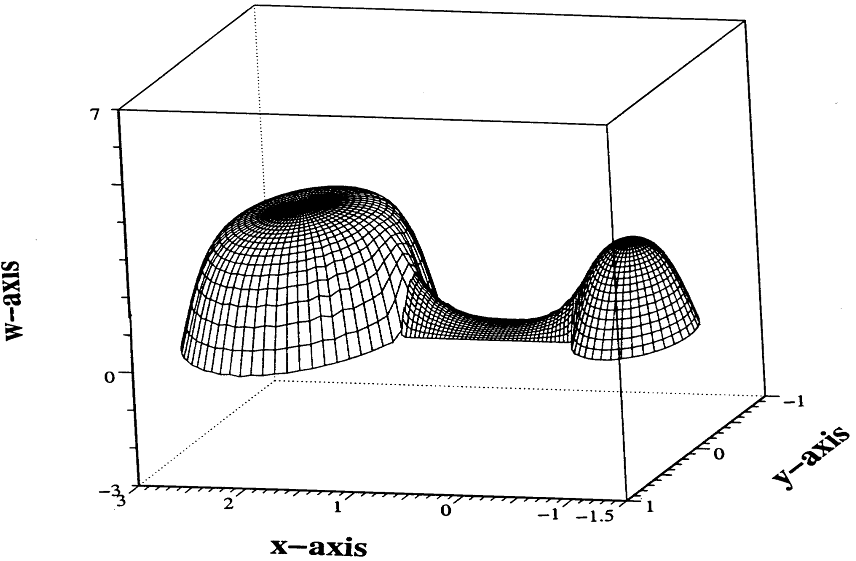

FIGURE 5. Global minimizer w, $\lambda = 50$, $J_\lambda(w) = -169.7$, $\max w = 3.119$.

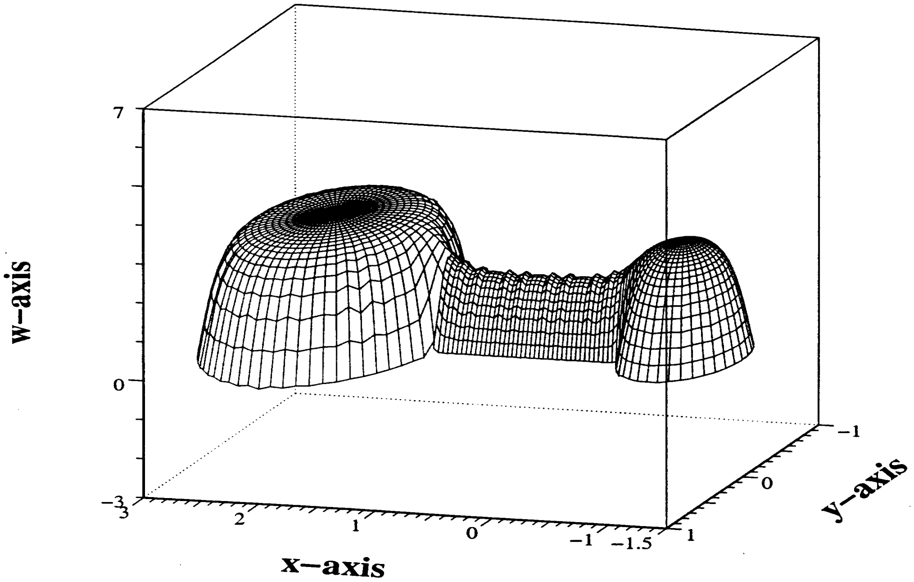

FIGURE 6. Global minimizer w, $\lambda = 100$, $J_\lambda(w) = -469.9$, $\max w = 3.14$.

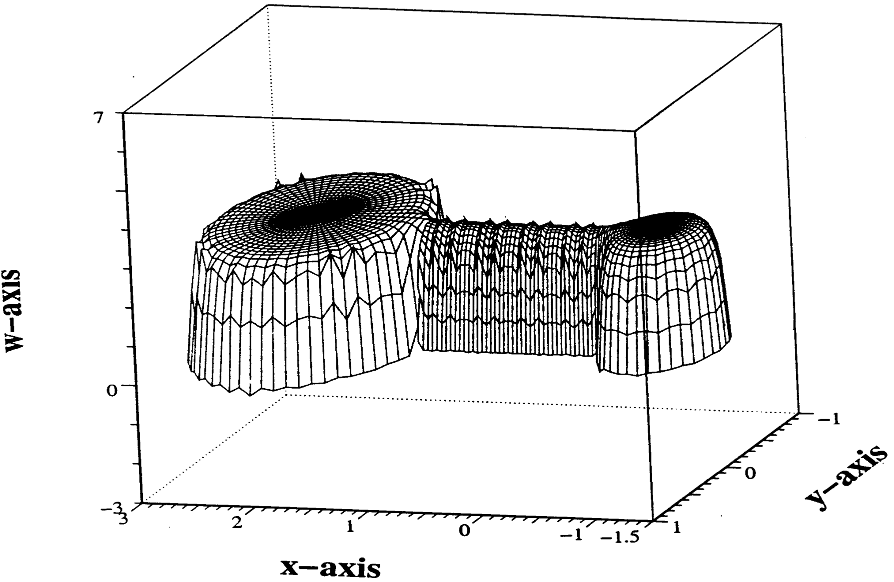

FIGURE 7. Global minimizer w, $\lambda = 500$, $J_\lambda(w) = -3506.7$, $\max w = 3.142$. Note that because of the large value of λ, numerical instability becomes noticeable.

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