Robust Boundary Control of the Stokes Fluids with Boundary Point Observations

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Abstract

In this paper, we first study an optimal boundary control problem governed by the Stokes system with point pressure observations on the boundary. A constrained LQR approach with some hydrostatic potential theory, boundary integral equations and a variational inequality in a Banach space setting is applied to establish a state feedback characterization of the optimal control. Since a hyper-singularity is involved in regularity/singularity analysis and the kernel is only p.v.-integrable, many Lebesgue integral related tools cannot be applied directly. We develop a method to handle such a hyper-singularity in establishing some new regularity results. Robust optimal control problems, i.e., to minimize the velocity at observation points while bringing down the pressure there, are then solved in the last section.

Keywords. LQR, Stokes fluid, distributed boundary control, point pressure observation, hydrostatic potential, BIE, hyper-singularity

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Abbreviated titles. ROBUST OPTIMAL CONTROL OF THE STOKES FLUIDS

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1 INTRODUCTION

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary Γ, and Ω be located on one side of Γ. The Stokes' equation

$$
(1.1)\qquad \begin{cases} \Delta\vec{w}(x) - \nabla p(x) = 0, & \text{in } \Omega, \\ \text{div } \vec{w}(x) = 0, & \text{in } \Omega, \\ \vec{\tau}(\vec{w})(x) = \vec{u}(x), & \text{on } \Gamma, \end{cases}
$$

describes the steady state of an incompressible viscous fluid with low velocity $\vec{w}(x)$ and pressure $p(x)$ in the domain Ω, where $\vec{\tau}(\vec{w})(x)$ is the surface stress of the fluid along the boundary Γ, given by

$$
\vec{\tau}(\vec{w})(x) = (\tau_1(\vec{w})(x), \tau_2(\vec{w})(x), \tau_3(\vec{w})(x))^T,
$$

$$
\tau_i(\vec{w})(x) = \sum_{k=1}^{3} \left[\frac{\partial w_i(x)}{\partial x_k} + \frac{\partial w_k(x)}{\partial x_i} \right] n_k(x) - p(x) n_i(x);
$$

$\vec{n}(x)$ is the unit outnormal vector of Γ at x and $\vec{u}(x)$ in a feasible control $\mathcal{U}$ is the (surface stress) Neumann boundary control on the surface Γ.

Let

$$
(1.2)\qquad M_0 = \{\vec{a} + \vec{b} \times \vec{x} \mid \vec{a}, \vec{b} \in \mathbb{R}^3\},
$$

which is the subspace of the rigid body motions in $\mathbb{R}^3$. Multiplying the Stokes equation by $\vec{a} + \vec{b} \times \vec{x} \in M_0$ and integration by parts yield the compatibility condition of the Stokes system, i.e.,

$$
\int_\Gamma \vec{\tau}(\vec{w})(x) \cdot (\vec{a} + \vec{b} \times \vec{x})\, d\sigma_x = 0, \qquad \text{or} \qquad \vec{\tau}(\vec{w}) \perp M_0.
$$

Linear-quadratic regulator approach for systems governed by partial differential equations has been widely used in research and applications, i.e., for given "target values" at observation locations, one tries to find a control such that the state variable is as close to the target values at observation locations as possible. From classic [4, 27] to recent works in [5,6,1,2, 22,23], most results in the literature assume that the target value is distributed in space. There are some results in the literature [4,27, 3] which assume that the target value is given pointwise, however only at some interior observation points. Therefore singularities can be avoided and Hilbert space setting can be used. By using a bounded linear operator, it can be treated as distributed target. So usual Galerkin variational approach can be applied and the optimal control can be characterized by an adjoint system.

In this paper, we assume that the interior of the domain is not reachable, any control of the system or observations of the state variable have to take place on the boundary. Basically there are two types of target values. For the first type, the target value is an ideal profile for the state variable, it can be either distributed or pointwise. For the second type, the target value is certain data for the state variable measured in experiment. In this case, assuming that the data is distributed in space is not realistic. In particular in a state-feedback control system, observations of the state variable can be realized by sensors as feedback to the control. For an example, [28,33,34] a low-cost fiber-optic Fabry-Perot sensor can be embedded to measure deformation, temperature, strain, pressure,..,etc. However those sensors can only detect *average* data in between the sensor and the size of the sensor can be less than 10^{-6}m. So point observations should be a better mathematical model. However, once point observations are used in a boundary control system, the state variable has to be continuous and singularities will appear in the system. A Banach space has to be used in the problem setting. Regularity/singularity analysis is the key to success. The usual Galerkin variational approach is not applicable and an adjoint system on the entire domain can only characterize the optimal control implicitly, which can not provide enough information on the singularity in the optimal control along the boundary, partially due to the difference in dimensionalities. We will adopt an approach in a Banch space setting, which combines boundary integral equations with hydrostatic potential theory and a variational inequality to derive a state-feedback characterization of the optimal control.

In [36], we studied optimal boundary control problems governed by the Stokes' equation with point velocity observations on the boundary, i.e., to find $\vec{u}(x) \in \mathcal{U}$ such that $\vec{w}(x)$ and $p(x)$ satisfy (1.1) and so as to

$$\min J(\vec{u}) = \sum_{k=1}^{m} \mu_k |\vec{w}(s^k) - \vec{Z}_k|^2 + \gamma \int_{\Gamma_1} |\vec{u}(x)|^2 d\sigma_x. \tag{1.3}$$

We established some regularity/singularity results and derived a velocity state feedback characterization of the optimal control from which we see that the optimal control oscillates between uper and lower bounds in any neighborhood of an observation point. Such an optimal control is hard to realize. It is also known that once velocity is maximized or minimized at certain points, pressure will build up there. It may cause problems to and even break the system. Such an optimal control is said to be not robust. So for the sake of robustness of the control system, pressure should also be taken into account. On the other hand, people may be just interested in controlling the pressure, so pressure observations should be taken into place. Since velocity is a vector and pressure is a scalar, pressure sensors are easier to design [28,33,34]. So we consider the following optimal boundary control problem governed by the Stokes' equation with boundary point pressure observations. Once the regularity/singularity analysis of this problem is done successfully, the robust control problem is just a combination of problems (1.3) and (1.4) and will be done in the last section.

Find $\vec{u}(x) \in \mathcal{U}$ such that $\vec{w}(x)$ and $p(x)$ satisfy (1.1) and so as to

$$\min J(\vec{u}) = \sum_{k=1}^{m} \mu_k |p(s^k) - Z_k|^2 + \gamma \int_{\Gamma_1} |\vec{u}(x)|^2 d\sigma_x, \tag{1.4}$$

where

$\mathcal{U}$ is the feasible constraint set be specified later for well-posedness of the problem;

$\gamma, \mu_k > 0,\ 1 \leq k \leq m$, are given weighting factors;

$s^k \in , , 1 \leq k \leq m$, are prescribed "observation points";

$Z_k, 1 \leq k \leq m$, are prescribed pressure "target values" at s^k.

Since point pressure observations have been used in the system, the pressure state variable $q(x)$ has to be continuous at observation points. Otherwise point observations will be meaningless. So the first question which has to be answered is that what space we should choose for the control variable $\vec{u}$ to ensure the continuity of the pressure at observation points. There is a trade-off. If the space we choose is too large, e.g.,the reflexive Banach space $L^p(,)$, then the pressure may fail to be continuous at observation points. On the other hand, if the space is too small, e.g., $C^{0,\alpha}(,)$—not a reflexive Banach space, then the pressure will be continuous at observation points, but we lost the weak compactness of the feasible control set $\mathcal{U}$. Consequently we fail to ensure the existence of the optimal control. So regularities of this problem has to be carefully analyzed and balanced.

Throughout of this paper, unless stated otherwise, we assume $p > 2, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$ and $\|\cdot\|$ is the norm in $(L^h(,\,))^n (h \geq 1)$ and $L^0(,\,)$ is the space of all p.v.-integrable functions on , , where p.v. stands for Cauchy principal value.

For $q \geq 1$, let A be a subspace of $(L^q(,\,))^3$ and denote

$$(L^q(,\,))^3_{\perp A} = \{\vec{f} \in (L^q(,\,))^3 | \vec{f} \perp A\}.$$

Let's briefly recall some hydrostatic potential theory and known regularity results.

Let $\{E(x,\xi), \vec{e}(x,\xi)\} = \{[E_{ij}(x,\xi)]_{3\times 3}, [e_i(x,\xi)]_{3\times 1}\}$ be the velocity-pressure fundamental solution of the Stokes systems, i.e.

$$\left\{ \begin{array}{rcl} \Delta_x E(x,\xi) - \nabla_x \vec{e}(x,\xi) & = & -\delta(x-\xi) I_3, \\ \mathrm{div}_x E(x,\xi) & = & 0 \end{array} \right. \tag{1.5}$$

where $\delta(x-\xi)$ is the unit Dirac delta function at $x = \xi$ and I_3 is the 3×3 identity matrix. It is well-known [26] that

$$\begin{aligned} E_{ij}(x,\xi) &= \frac{1}{8\pi}\left(\frac{\delta_{ij}}{|x-\xi|} + \frac{(x_i-\xi_i)(x_j-\xi_j)}{|x-\xi|^3}\right), \quad 1 \leq i,j \leq 3, \\ e_i(x,\xi) &= \frac{1}{4\pi}\frac{x_i-\xi_i}{|x-\xi|^3}, \end{aligned}$$

where $\delta_{i,j}$ is the Kronecker symbol.

Remark 1.1 Here we point out some significant differences among the cases with point velocity observations, point pressure observations, distributed velocity observations and distributed pressure observations. For a given vector $\vec{V} \in \mathbb{R}^3$ the function

$$x \mapsto \sum_{k=1}^{m} \mu_k E(s^k, x)\vec{V} \tag{1.6}$$

has a matrix (weak) singularity of order $O(\frac{1}{|x-s^k|})$ at $x = s^k$ and the function

$$x \mapsto \sum_{k=1}^{m} \mu_k \vec{e}(s^k, x) \cdot \vec{V} \tag{1.7}$$

has a vector (hyper) singularity of order $O(\frac{1}{|x-s^k|^2})$ at $x = s^k$. They both may oscillate between $-\infty$ and $+\infty$ as $x \to s^k$, so they are very tough to deal with. On the other hand, the function

$$x \mapsto \int_\Gamma E(\xi, x)\vec{V}(\xi) d\sigma_\xi \tag{1.8}$$

is well-defined and continuous for any $\vec{V} \in L^2(,\ )$. But the nature of the function

$$x \mapsto \int_\Gamma \vec{e}(\xi, x)\vec{V}(\xi) d\sigma_\xi \tag{1.9}$$

is not yet clear. Even for $\vec{V} \equiv 1$, the integral is not Lebesgue integrable but integrable only in the sense of Cauchy principal value. Consequently, we cannot take absolute value, there is no norm defined and many well-known Lebesgue integral related theorems and inequalities, such as the Fubini theorem, the Lebesgue dominant convergence theorem and the Holder inequality, etc., cannot be applied directly. Therefore regularity/singularity analysis becomes very difficult.

It is known that the solution $(\vec{w}, p)$ of the Stokes equation (1.1) has a simple-layer representation

$$\vec{w}(x) = \int_\Gamma E(x, \xi)\vec{\eta}(\xi) d\sigma_\xi + \vec{a} + \vec{b} \times \vec{x} \quad \forall\, x \in \overline{\Omega}, \tag{1.10}$$

$$p(x) = \int_\Gamma \vec{e}(x, \xi) \cdot \vec{\eta}(\xi) d\sigma_\xi + a \quad \forall\, x \in \Omega, \tag{1.11}$$

$$p(x) = \frac{1}{2}\vec{n}(x) \cdot \vec{\eta}(x) + \int_\Gamma \vec{e}(x, \xi) \cdot \vec{\eta}(\xi) d\sigma_\xi + a \quad \forall\, x \in ,\ , \tag{1.12}$$

with the layer density $\vec{\eta}$ and for some constants $\vec{a}, \vec{b} \in \mathbb{R}^3$ and $a \in \mathbb{R}$. $\vec{a} + \vec{b} \times \vec{x}$ represents a rigid body motion. By the jump property of the layer potentials, we obtain the boundary integral equation

$$\vec{\tau}(\vec{w})(x) = \frac{1}{2}\vec{\eta}(x) + \text{p.v.} \int_\Gamma T(x, \xi)\vec{\eta}(\xi) d\sigma_\xi \quad \forall\, x \in ,\ , \tag{1.13}$$

where

$$T(x, \xi) = [\vec{\tau}_x(E_1)(x, \xi), \vec{\tau}_x(E_2)(x, \xi), \vec{\tau}_x(E_3)(x, \xi)] = [T_{ij}(x, \xi)]_{3\times 3},$$
$$T_{ij}(x, \xi) = -\frac{3}{4\pi} \frac{(x_i - \xi_i)(x_j - \xi_j)}{|x - \xi|^5} (x - \xi) \cdot \vec{n}(x).$$

With a given Neumann B.D., the layer density $\vec{\eta}$ can be solved from the above BIE (1.13). Once the layer density is known, the solution $(\vec{w}(x), p(x))$ can be computed from (1.10) and (1.11).

With a boundary element method (BEM), the boundary , is divided into N elements with nodal points x_i. Assume that the layer density $\vec{\eta}(x)$ is piecewise smooth, e.g. piecewise constant, piecewise linear,$\cdots$, etc., then the BIE (1.13) becomes a linear algebraic system. This system can be solved for $\vec{\eta}(x_i)$ and then $(\vec{w}(x), p(x))$ can be computed from a discretized version of (1.10), (1.11) and (1.12) for any $x \in \overline{\Omega}$.

For each $\vec{f} \in (L^2(,\ ))^3$ and $x \in \mathbb{R}^3$, we define the simple layer velocity potential $\mathcal{S}_v(\vec{f})$ and simple-layer pressure potential $\mathcal{S}_p(\vec{f})$ by

$$\begin{aligned} \mathcal{S}_v(\vec{f})(x) &= \int_\Gamma E(x,\xi)\vec{f}(\xi)\, d\sigma_\xi, \\ \mathcal{S}_p(\vec{f})(x) &= \int_\Gamma \vec{e}(x,\xi)\cdot\vec{f}(\xi)\ \ d\sigma_\xi. \end{aligned}$$

For each $\vec{f} \in (L^2(,\ ))^3$ and $x \in ,$, we define the boundary operators $\mathcal{K}$ and $\mathcal{K}^*$ by

$$\begin{aligned} \mathcal{K}(\vec{f})(x) &= \text{p.v.} \int_\Gamma Q(x,\xi)\vec{f}(\xi) d\sigma_\xi \\ \mathcal{K}^*(\vec{f})(x) &= \text{p.v.} \int_\Gamma T(x,\xi)\vec{f}(\xi) d\sigma_\xi, \end{aligned}$$

where

$$\begin{gathered} Q(x,\xi) = [\vec{\tau}_\xi(E_1)(x,\xi), \vec{\tau}_\xi(E_2)(x,\xi), \vec{\tau}_\xi(E_3)(x,\xi)] = [Q_{ij}(x,\xi)]_{3\times 3}, \\ Q_{ij}(x,\xi) = \frac{3}{4\pi}\frac{(x_i-\xi_i)(x_j-\xi_j)}{|x-\xi|^5}(x-\xi)\cdot\vec{n}(\xi). \end{gathered}$$

For a proper given Neumann B.D. $\vec{u}$, the Stokes system (1.1) has a velocity solution $\vec{w}$ unique up to a vector $\vec{a} + \vec{b}\times\vec{x} \in M_0$ and a pressure solution $p(x)$ unique up to a constant, i.e.,

$$\vec{w}(x,\vec{u}) = \mathcal{S}_v \circ (\frac{1}{2}I + \mathcal{K}^*)^{-1}(\vec{u})(x) + \vec{a} + \vec{b}\times x, \quad x \in \overline{\Omega}, \tag{1.14}$$

$$p(x,\vec{u}) = \mathcal{S}_p \circ (\frac{1}{2}I + \mathcal{K}^*)^{-1}(\vec{u})(x) + a, \quad x \in \Omega, \tag{1.15}$$

$$p(x,\vec{u}) = (\frac{1}{2}\vec{n}(x) + \mathcal{S}_p) \circ (\frac{1}{2}I + \mathcal{K}^*)^{-1}(\vec{u})(x) + a, \quad x \in ,\ . \tag{1.16}$$

With

$$\vec{w}_0 = \mathcal{S}_v \circ (\frac{1}{2}I + \mathcal{K}^*)^{-1}(\vec{u}),$$

$$p_0 = (\frac{1}{2}\vec{n}(x) + \mathcal{S}_p) \circ (\frac{1}{2}I + \mathcal{K}^*)^{-1}(\vec{u}),$$

we may write

$$\begin{aligned} \vec{w}(x) &= \vec{w}_0(x) + \vec{a} + \vec{b} \times x, \quad x \in \overline{\Omega}, \\ p(x) &= p_0(x) + a, \quad x \in , . \end{aligned}$$

Let

$$N = \ker\, (\frac{1}{2}I + \mathcal{K}^*),$$

which represents the set of all layer densities corresponding to the zero Neumann B.D., with which the Stokes system has only a rigid body motion and a constant pressure. Hence we have

$$M_0 = \mathcal{S}_v(N) = \ker\, (\frac{1}{2}I + \mathcal{K}), \tag{1.17}$$

$$\mathcal{S}_p(N) = \mathbb{R}.$$

2 Some Inequalities and Regularities

To study regularity of the pressure solution $p(x, \vec{u})$, we first collect some known regularity results on $\mathcal{S}_v, \mathcal{S}_p, \mathcal{K}$ and $\mathcal{K}^*$ into a lemma.

Lemma 2.1 *[36] Let $\Omega \subset \mathbb{R}^3$ be a bounded simply connected domain with smooth boundary , . Let $\mathcal{K}'$ denote either $\mathcal{K}$ or $\mathcal{K}^*$,*

(a) $\mathcal{S}_v : (L^p(,))^3 \mapsto (C^{0,\alpha}(\mathbb{R}^3)^3)$ is a linear operator for $p > 2$ and $0 < \alpha < \frac{p-2}{p}$;

(b) For any $1 < p < +\infty$, $\mathcal{K}' : (L^p(,))^3 \mapsto (L^p(,))^3$ is a linear compact operator and $\mathcal{K}$ ($\mathcal{K}^*$) is the adjoint of $\mathcal{K}^*$ ($\mathcal{K}$);

(c) For $p > 2$ and $0 < \alpha < \frac{p-2}{p}$, $\mathcal{K}' : (L^p(,))^3 \mapsto (C^{0,\alpha}(,))^3$ is a bounded linear operator;

(d) For $1 < p < \infty$,

(1) $(\frac{1}{2}I + \mathcal{K}) : (L^p(,))^3_{\perp M_0} \mapsto (L^p(,))^3_{\perp N}$ is invertible;

(2) $(\frac{1}{2}I + \mathcal{K}^*) : (L^p(,))^3_{\perp M_0} \mapsto (L^p(,))^3_{\perp M_0}$ is invertible;

(e) *For* $1 < q < 2$ *and* $s < \frac{2q}{2-q}$, $\mathcal{K}' : \ (L^q(,))^3 \mapsto (L^s(,))^3$ *is a bounded linear operator. Therefore* $\mathcal{K}' \circ \mathcal{K}' : (L^q(,))^3 \mapsto (C^{0,\alpha}(,))^3$ *for every* $q > 1$ *and* $0 < \alpha < \frac{q-1}{q}$;

(f) *For* $0 < \alpha \leq 1$,

(1) $(\frac{1}{2}I + \mathcal{K}) : \ (C^{0,\alpha}(,))^3_{\perp M_0} \mapsto (C^{0,\alpha}(,))^3_{\perp N}$ *is invertible.*

(2) $(\frac{1}{2}I + \mathcal{K}^*) : \ (C^{0,\alpha}(,))^3_{\perp M_0} \mapsto (C^{0,\alpha}(,))^3_{\perp M_0}$ *is invertible.*

Since $\mathcal{K}, \mathcal{K}^*$ are compact,

$$\text{ind}\,(\frac{1}{2}I + \mathcal{K}^*) = \text{ind}\,(\frac{1}{2}I + \mathcal{K}) = 0.$$

We have

$$\dim\, N = \dim\, \ker\,(\frac{1}{2}I + \mathcal{K}^*) = \text{codim}\,\mathcal{R}(\frac{1}{2}I + \mathcal{K}^*) = \dim\, M_0 = 6.$$

By the Fubini theorem and the adjoint relation between $\mathcal{K}$ and $\mathcal{K}^*$, we see

$$\begin{aligned} h \in (L^q(,))^3_{\perp N} \ &\mapsto \ \mathcal{K}(h) \in (L^q(,))^3_{\perp N}; \\ h \in (L^q(,))^3_{\perp M_0} \ &\mapsto \ \mathcal{K}^*(h) \in (L^q(,))^3_{\perp M_0}. \end{aligned}$$

Also it is easy to check that for each $\vec{u} \in (L^q(,))^3_{\perp M_0}$ and $\vec{h} \in (L^q(,))^3_{\perp N}$ we have

$$(\frac{1}{2}I + \mathcal{K}^*)^{-1}(\vec{u})(x) = 2\vec{u}(x) - 2(\frac{1}{2}I + \mathcal{K}^*)^{-1}\mathcal{K}^*(\vec{u})(x), \tag{2.1}$$

$$(\frac{1}{2}I + \mathcal{K})^{-1}(\vec{h})(x) = 2\vec{h}(x) - 2(\frac{1}{2}I + \mathcal{K})^{-1}\mathcal{K}(\vec{h})(x) + \vec{a} + \vec{b} \times x, \tag{2.2}$$

where $\vec{a} + \vec{b} \times x \in M_0$ is given by Theorem 2.2, s.t. $2\vec{h}(x) + \vec{a} + \vec{b} \times x \perp M_0$. Since the operator $\mathcal{K}^*$ somewhat increases regularity, the above decomposition formula implies that $\vec{u}$ and $(\frac{1}{2}I + \mathcal{K}^*)^{-1}(\vec{u})$ have the same regularities. This observation will help us in regularity analysis of the pressure state.

The regularity analysis of the operator $\mathcal{S}_p$ is much tougher, because it involves a hyper-singular kernel which is integrable only in the sense of Cauchy principal value. A method will be developed to handle such a hyper-singularity. The basic idea is that to establish the regularity of a term containing a hyper-singularity, we first find a term with the same known-regularity such that the difference of those two terms is Lebesgue integrable, so we can take absolute value or apply certain Lebesgue integral related theorems and inequalities. The following inequalities will be frequently used.

Lemma 2.2

$$(2.3)\qquad \int_\Gamma \frac{d\sigma_\xi}{|x-\xi|^\alpha |y-\xi|^\beta} \le \begin{cases} C(,\,,\alpha,\beta), & \text{if } 0<2-\alpha-\beta, \\ C(,\,,\alpha,\beta,\gamma)|x-y|^\gamma, & \text{if } 0\ge 2-\alpha-\beta>\gamma, \\ C(,\,,\alpha,\beta)|x-y|^{2-\alpha-\beta}, & \text{if } 0>2-\alpha-\beta, \end{cases}$$

for any $\alpha, \beta < 2$;

$$(2.4)\qquad \left|\int_\Gamma \frac{1}{|x-\xi|^\alpha}\frac{y_i-\xi_i}{|y-\xi|^3}\,d\sigma_\xi\right| \le \begin{cases} C(,\,), & \text{if } \alpha=0, \\ C(,\,,\alpha,\gamma)|x-y|^{-\gamma}, & \text{if } 0\le\alpha<\gamma<2; \end{cases}$$

$$(2.5)\qquad \left|\int_\Gamma \frac{(x-\xi)\cdot\vec{n}(\xi)}{|x-\xi|^3}\frac{y_i-\xi_i}{|y-\xi|^3}\,d\sigma_\xi\right| \le \frac{C(,\,,\alpha)}{|x-y|^\alpha},\ \text{for } 0\le\alpha<2.$$

Proof. (2.3) is established in [38]. To prove (2.4), first for $0<\alpha<\gamma\le 1$, we have

$$\begin{aligned}
&\left|\int_\Gamma \frac{1}{|x-\xi|^\alpha}\frac{y_i-\xi_i}{|y-\xi|^3}\,d\sigma_\xi\right| \\
\le\ & \left|\int_\Gamma \left[\frac{1}{|x-\xi|^\alpha}-\frac{1}{|x-y|^\alpha}\right]\frac{y_i-\xi_i}{|y-\xi|^3}\,d\sigma\right| + \frac{1}{|x-y|^\alpha}\left|\int_\Gamma \frac{y_i-\xi_i}{|y-\xi|^3}\,d\sigma\right| \\
\le\ & \int_\Gamma \frac{\big||x-y|^\alpha-|x-\xi|^\alpha\big|}{|x-\xi|^\alpha|x-y|^\alpha}\frac{1}{|y-\xi|^2}\,d\sigma_\xi + \frac{C_1(,\,)}{|x-y|^\alpha} \\
\le\ & \int_\Gamma \frac{|y-\xi|^\alpha}{|x-\xi|^\alpha|x-y|^\alpha}\ \frac{1}{|y-\xi|^2}\,d\sigma_\xi + \frac{C_1(,\,,\alpha,\gamma)}{|x-y|^\gamma} \\
=\ & \frac{1}{|x-y|^\alpha}\int_\Gamma \frac{1}{|x-\xi|^\alpha}\frac{1}{|y-\xi|^{2-\alpha}}\,d\sigma_\xi + \frac{C_1(,\,,\alpha,\gamma)}{|x-y|^\gamma} \quad \text{(by (2.3))} \\
\le\ & \frac{1}{|x-y|^\alpha}\frac{C_2(,\,,\alpha,\gamma)}{|x-y|^{\gamma-\alpha}} + \frac{C_1(,\,,\alpha,\gamma)}{|x-y|^\gamma} \\
=\ & \frac{C(,\,,\alpha,\gamma)}{|x-y|^\gamma}.
\end{aligned}$$

Next for $1\le\alpha<\gamma<2$, we have

$$\begin{aligned}
&\left|\int_\Gamma \frac{1}{|x-\xi|^\alpha}\frac{y_i-\xi_i}{|y-\xi|^3}\,d\sigma_\xi\right| \\
\le\ & \int_\Gamma \left|\frac{1}{|x-\xi|}-\frac{1}{|x-y|}\right|\frac{1}{|x-\xi|^{\alpha-1}}\frac{1}{|y-\xi|^2}\,d\sigma_\xi + \frac{1}{|x-y|}\left|\int_\Gamma \frac{1}{|x-\xi|^{\alpha-1}}\frac{y_i-\xi_i}{|y-\xi|^3}\,d\sigma_\xi\right| \\
\le\ & \int_\Gamma \frac{|y-\xi|}{|x-\xi||x-y|}\frac{1}{|x-\xi|^{\alpha-1}}\frac{1}{|y-\xi|^2}\,d\sigma_\xi + \frac{C_1(,\,,\alpha,\gamma)}{|x-y||x-y|^{\gamma-\alpha+(\alpha-1)}} \\
\le\ & \frac{1}{|x-y|}\int_\Gamma \frac{1}{|x-\xi|^\alpha}\frac{1}{|y-\xi|}\,d\sigma_\xi + \frac{C_1(,\,,\alpha,\gamma)}{|x-y|^\gamma} \\
\le\ & \frac{C(,\,,\alpha,\gamma)}{|x-y|^\gamma},
\end{aligned}$$

where in the last two inequalities, we have used (2.4) for $0 < \alpha < \gamma \leq 1$ and (2.3). To prove (2.5), we use Lemma 3.4.6 in [17], .i.e,

$$\left|\frac{\langle \vec{n}(\xi), x-\xi\rangle}{|x-\xi|^3} - \frac{\langle \vec{n}(y), x-y\rangle}{|x-y|^3}\right| \leq C|y-\xi|^{\alpha}\left\{\frac{1}{|x-\xi|^{1+\alpha}} + \frac{1}{|x-y|^{1+\alpha}}\right\}, \tag{2.6}$$

and the inequality

$$|(x-y)\cdot\vec{n}(y)| \leq C'|x-y|^2,$$

where the constant C' is independent of x, y. We see

$$\begin{aligned}
&\left|\int_{\Gamma}\frac{(x-\xi)\cdot\vec{n}(\xi)}{|x-\xi|^3}\frac{y_i-\xi_i}{|y-\xi|^3}\,d\sigma_\xi\right| \\
= &\left|\int_{\Gamma}\left[\frac{(x-\xi)\cdot\vec{n}(\xi)}{|x-\xi|^3} - \frac{(x-y)\cdot\vec{n}(y)}{|x-y|^3} + \frac{(x-y)\cdot\vec{n}(y)}{|x-y|^3}\right]\frac{y_i-\xi_i}{|y-\xi|^3}\,d\sigma_\xi\right| \\
\leq &\int_{\Gamma} C|y-\xi|^{\alpha}\left\{\frac{1}{|x-\xi|^{1+\alpha}} + \frac{1}{|x-y|^{1+\alpha}}\right\}\frac{1}{|y-\xi|^2}\,d\sigma_\xi + \left|\frac{(x-y)\cdot\vec{n}(y)}{|x-y|^3}\int_{\Gamma}\frac{y_i-\xi_i}{|y-\xi|^3}\,d\sigma_\xi\right| \\
\leq &\; C\int_{\Gamma}\frac{1}{|x-\xi|^{1+\alpha}}\frac{1}{|y-\xi|^{2-\alpha}}\,d\sigma_\xi + \frac{C}{|x-y|^{1+\alpha}}\int_{\Gamma}\frac{1}{|y-\xi|^{2-\alpha}}\,d\sigma_\xi + \frac{C'}{|x-y|} \\
\leq &\;\frac{C_1}{|x-y|} + \frac{C_2}{|x-y|^{1+\alpha}} + \frac{C'}{|x-y|} \\
\leq &\;\frac{C_0}{|x-y|^{1+\alpha}}
\end{aligned}$$

where in the above, we have applied inequalities (2.6), (2.4) and (2.3). ∎

The following is a key lemma in our regularity analysis of the operator $\mathcal{S}_p$ and the pressure state.

Lemma 2.3 *The mapping*

$$x \mapsto \int_{\Gamma}\vec{e}(x,\xi)\,d\sigma_\xi \in (C^1(,\,))^3.$$

Proof. For each $x = (x_1, x_2, x_3) \in\,,$, we denote the outward normal vector $\vec{n}(x) = (n_1(x), n_2(x), n_3(x))$, $\bar{x} = (x_1, x_2)$ and $,_c(x,r) = \{\xi \in\,, ||\bar{\xi} - \bar{x}| < r\}$. Since

$$d\sigma_\xi = \frac{d\xi_1\,d\xi_2}{\langle \vec{n}(x), \vec{n}(\xi)\rangle}$$

and

$$\int_{|\bar{\xi}-\bar{x}|<r}\frac{x_1-\xi_1}{|\bar{x}-\bar{\xi}|^3}\,d\xi_1\,d\xi_2 = 0,$$

we see that

$$\int_{\Gamma} \frac{x_1 - \xi_1}{|\bar{x} - \bar{\xi}|^3} \langle \vec{n}(x), \vec{n}(\xi) \rangle \, d\sigma_\xi = \int_{\Gamma \setminus \Gamma_c(x,r)} \frac{x_1 - \xi_1}{|\bar{x} - \bar{\xi}|^3} \langle \vec{n}(x), \vec{n}(\xi) \rangle \, d\sigma_\xi$$

contains no singularity and therefore is in $(C^\infty(\,,\,))^3$. So it remains to show that

$$\int_{\Gamma} G(x, \xi) \, d\sigma_\xi \in C^1(\,,\,)$$

where

$$G(x, \xi) = \frac{x_1 - \xi_1}{|x - \xi|^3} - \frac{x_1 - \xi_1}{|\bar{x} - \bar{\xi}|^3} \langle \vec{n}(x), \vec{n}(\xi) \rangle.$$

To verify this, by the Lebesgue dominated convergence theorem, we only have to show that

$$\int_{\Gamma} |G(x, \xi)| \, d\sigma_\xi \in C(\,,\,) \quad \text{and} \quad \int_{\Gamma} |G_{x_i}(x, \xi)| \, d\sigma_\xi \in C(\,,\,),$$

for $i = 1, 2, 3$, or equivalently, we can show that

$$|G(x, \xi)| \leq M' \quad \text{and} \quad |G_{x_i}(x, \xi)| \leq \frac{M_1}{|\bar{x} - \bar{\xi}|}$$

for $i = 1, 2, 3$ and some constants M' and M_1.

Since , is smooth, there is a local rectangular coordinate system around x, i.e., there is $\phi \in C^\infty(\mathbb{R}^2)$ and $r > 0$ such that

$$\xi = (\xi_1, \xi_2, \phi(\xi_1, \xi_2)), \quad \forall\, \xi \in\, ,\, _c(x, r)$$

with

$$\phi_{\xi_1}(\bar{x}) = \phi_{\xi_2}(\bar{x}) = 0.$$

Then we have

(2.7) $$\phi(\xi_1, \xi_2) = x_3 + R_1(\bar{x}, \bar{\xi}),$$

(2.8) $$\langle \bar{n}(x), \bar{n}(\xi) \rangle = 1 + R_2(\bar{x}, \bar{\xi})$$

with

(2.9) $$|R_1(\bar{x}, \bar{\xi})| \leq M|\bar{x} - \bar{\xi}|^2 \quad \text{and} \quad |R_2(\bar{x}, \bar{\xi})| \leq M|\bar{x} - \bar{\xi}|^2$$

for some constant $M > 0$. Notice that

$$\begin{aligned} \frac{1}{(1+z^2)^{\frac{k}{2}}} &= 1 + \frac{1-(1+z^2)^{\frac{k}{2}}}{(1+z^2)^{\frac{k}{2}}} \\ &= 1 + \frac{1-(1+z^2)^k}{(1+z^2)^{\frac{k}{2}}(1+(1+z^2)^{\frac{k}{2}})} \\ &\equiv 1 + H_k(z) \end{aligned}$$

with

$$|H_k(z)| \leq M|z|^2$$

for some constant $M > 0$. Now for $z = \frac{R_1(\bar{x},\bar{\xi})}{|\bar{x}-\bar{\xi}|}$, we see

$$\begin{aligned} \frac{1}{|x-\xi|^k} &= \frac{1}{|\bar{x}-\bar{\xi}|^k}\frac{1}{(1+z^2)^{\frac{k}{2}}} \\ &= \frac{1}{|\bar{x}-\bar{\xi}|^k}(1+H_k(\bar{x},\bar{\xi})) \end{aligned}$$

where

$$|H_k(\bar{x},\bar{\xi})| = |H_k(z)| \leq M|\bar{x}-\bar{\xi}|^2.$$

Since

$$\frac{x_1-\xi_1}{|x-\xi|^3} = \frac{x_1-\xi_1}{|\bar{x}-\bar{\xi}|^3}(1+H_3(\bar{x},\bar{\xi})),$$

$$\frac{x_1-\xi_1}{|\bar{x}-\bar{\xi}|^3}\langle \bar{n}(x), \bar{n}(\xi)\rangle = \frac{x_1-\xi_1}{|\bar{x}-\bar{\xi}|^3}(1+R_2(\bar{x},\bar{\xi})),$$

we see that

$$\begin{aligned} |G(x,\xi)| &\leq \frac{|x_1-\xi_1|}{|\bar{x}-\bar{\xi}|^3}(|H_3(\bar{x},\bar{\xi})| + |R_2(\bar{x},\bar{\xi})|) \\ &\leq M'. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\frac{\partial}{\partial x_1}\frac{x_1-\xi_1}{|x-\xi|^3} \\ =\ & \frac{|x-\xi|^3 - 3(x_1-\xi_1)^2}{|x-\xi|^5} \\ =\ & \frac{(x_2-\xi_2)^2 - 2(x_1-\xi_1)^2}{|x-\xi|^5} + \frac{(x_3-\xi_3)^2}{|x-\xi|^5} \\ =\ & \frac{(x_2-\xi_2)^2 - 2(x_1-\xi_1)^2}{|\bar{x}-\bar{\xi}|^5}(1+H_5(\bar{x},\bar{\xi})) + \frac{(x_3-\xi_3)^2}{|x-\xi|^5}; \end{aligned}$$

$$\begin{aligned}
&\frac{\partial}{\partial x_1}\left(\frac{x_1-\xi_1}{|\bar{x}-\bar{\xi}|^3}\langle \bar{n}(x), \bar{n}(\xi)\rangle\right) \\
=\ & \frac{(x_2-\xi_2)^2-2(x_1-\xi_1)^2}{|\bar{x}-\bar{\xi}|^5}\langle \bar{n}(x), \bar{n}(\xi)\rangle + \frac{x_1-\xi_1}{|\bar{x}-\bar{\xi}|^3}\langle \frac{\partial}{\partial x_1}\bar{n}(x), \bar{n}(\xi)\rangle \\
=\ & \frac{(x_2-\xi_2)^2-2(x_1-\xi_1)^2}{|\bar{x}-\bar{\xi}|^5}(1+R_2(\bar{x},\bar{\xi})) + \frac{x_1-\xi_1}{|\bar{x}-\bar{\xi}|^3}\langle \frac{\partial}{\partial x_1}\bar{n}(x), \bar{n}(\xi)\rangle.
\end{aligned}$$

Since

$$\frac{(x_3-\xi_3)^2}{|x-\xi|^5} \leq \left|\frac{R_1^2(\bar{x},\bar{\xi})}{|\bar{x}-\bar{\xi}|^5}(1+H_5(\bar{x},\bar{\xi}))\right| \leq \frac{M}{|\bar{x}-\bar{\xi}|}$$

and from (2.8) and (2.9)

$$|\langle \frac{\partial}{\partial x_1}\bar{n}(x), \bar{n}(\xi)\rangle| \leq M|\bar{x}-\bar{\xi}|,$$

we have

$$\begin{aligned}
&|G_{x_1}(x,\xi)| \\
=\ & \left|\frac{(x_2-\xi_2)^2-2(x_1-\xi_1)^2}{|\bar{x}-\bar{\xi}|^5}(H_5(\bar{x},\bar{\xi})-R_2(\bar{x},\bar{x})) + \frac{(x_3-\xi_3)^2}{|x-\xi|^5} + \frac{x_1-\xi_1}{|\bar{x}-\bar{\xi}|^3}\langle \frac{\partial}{\partial x_1}\bar{n}(x), \bar{n}(\xi)\rangle\right| \\
\leq\ & \frac{M_1}{|\bar{x}-\bar{\xi}|}.
\end{aligned}$$

Similarly for $i = 2, 3$,

$$\begin{aligned}
\frac{\partial}{\partial x_i}\frac{x_1-\xi_1}{|x-\xi|^3} &= -3\frac{(x_1-\xi_1)(x_i-\xi_i)}{|\bar{x}-\bar{\xi}|^5}(1+H_5(\bar{x},\bar{\xi})); \\
\frac{\partial}{\partial x_i}\Big[\frac{x_1-\xi_1}{|\bar{x}-\bar{\xi}|^3}\langle \bar{n}(x), \bar{n}(\xi)\rangle\Big] &= -3\frac{(x_1-\xi_1)(x_i-\xi_i)}{|\bar{x}-\bar{\xi}|^5}(1+R_2(\bar{x},\bar{\xi})) + \frac{x_1-\xi_1}{|\bar{x}-\bar{\xi}|^3}\langle \frac{\partial}{\partial x_i}\bar{n}(x), \bar{n}(\xi)\rangle
\end{aligned}$$

lead to

$$|G_{x_i}(x,\xi)| \leq \frac{M_1}{|\bar{x}-\bar{\xi}|}.$$

The proof is completed. ▮

Lemma 2.4 *The pressure potential* $\mathcal{S}_p$ *defines a bounded linear operator from* $(C^{0,\alpha}(,\ ))^3$ *into* $C^{0,\alpha}(,\ )$ *for some* $0 < \alpha < 1$.

Proof. It is sufficient to show that

$$\int_\Gamma e_i(x,\xi) f(\xi)\, d\sigma_\xi \in C^{0,\alpha}(,\ )$$

for every $f \in C^{0,\alpha}(,\ )$. Notice

$$|e_i(x,\xi)| \leq |x-\xi|^{-2}$$

and for $2|x-x'| \leq |x-\xi|$, $\xi, x, x' \in ,$, we have

$$\begin{aligned} |x'-\xi| &\leq |x-x'|+|x'-\xi| \leq \frac{3}{2}|x-\xi|, \\ |x'-\xi| &\geq |x-\xi|-|x-\xi| \geq \frac{1}{2}|x-\xi| \end{aligned}$$

and

$$\begin{aligned} & |e_i(x,\xi)-e_i(x',\xi)| \\ = & \left| \frac{x_i-\xi_i}{|x-\xi|^3} - \frac{x'_i-\xi_i}{|x'-\xi|^3} \right| \\ \leq & \frac{|x_i-x'_i|}{|x-\xi|^3} + |x'_i-\xi_i| \left| \frac{1}{|x-\xi|^3} - \frac{1}{|x'-\xi|^3} \right| \\ \leq & |x-x'||x-\xi|^{-3} + |x'-\xi|\,||x-\xi|-|x'-\xi||\max\left(\frac{1}{|x-\xi|^4}, \frac{1}{|x'-\xi|^4}\right) \\ \leq & |x-x'||x-\xi|^{-3} + \frac{3}{2}|x-\xi||x-x'|\cdot|x-\xi|^{-4} \\ = & 25|x-x'|\cdot|x-\xi|^{-3}. \end{aligned} \tag{2.10}$$

It is also clear that

$$\left| \int_{(\xi\in\Gamma)\cap(|x-\xi|\geq r)} e_i(x,\xi)\, d\sigma_\xi \right| \leq M < \infty.$$

So all the conditions of Lemma 6.5.3 in [7] are satisfied, thus

$$\int_\Gamma e_i(x,\xi)(f(\xi)-f(x))\, d\sigma_\xi$$

defines a bounded linear operator from $C^{0,\alpha}(,\ )$ into $C^{0,\alpha}(,\ )$. Now

$$\int_\Gamma e_i(x,\xi)f(\xi)\, d\sigma_\xi = \int_\Gamma e_i(x,\xi)(f(\xi)-f(x))\, d\sigma_\xi + f(x)\int_\Gamma e_i(x,\xi)\, d\sigma_\xi,$$

take Lemma 2.3 into account, we conclude that

$$f \mapsto \int_\Gamma e_i(\cdot,\xi)f(\xi)\, d\sigma_\xi$$

defines a bounded linear operator from $C^{0,\alpha}(,\ )$ into $C^{0,\alpha}(,\ )$ for some $0<\alpha<1$. ∎

Theorem 2.1 *For each* $\vec{u} \in (C^{0,\alpha}(,\ ))^3_{\perp M_0}$, *the pressure* $p \in C^{0,\alpha}(,\ )$.

Proof. We have

$$p(x,\vec{u}) = \frac{1}{2}\vec{n}(x)\cdot(\frac{1}{2}I+\mathcal{K}^*)^{-1}(\vec{u})(x) + \mathcal{S}_p \circ (\frac{1}{2}I+\mathcal{K}^*)^{-1}(\vec{u})(x) + a, \quad x \in ,\ .$$

By Lemma 2.1 (f),

$$(\frac{1}{2}I + \mathcal{K}^*)^{-1}(\vec{u}) \in (C^{0,\alpha}(,\ ))^3.$$

Lemma 2.4 implies that

$$\mathcal{S}_p \circ (\frac{1}{2}I + \mathcal{K}^*)^{-1}(\vec{u}) \in C^{0,\alpha}(,\ ).$$

Finally , is C^2, we have

$$|\vec{n}(x) - \vec{n}(y)| \leq M|x - y|,$$

so

$$\frac{1}{2}\vec{n} \cdot (\frac{1}{2}I + \mathcal{K}^*)^{-1}(\vec{u}) \in C^{0,\alpha}(,\ ).$$

∎

However if we let the feasible control set $\mathcal{U} = C^{0,\alpha}(,\ ))^3_{\perp M_0}$, a non-reflexive Banach space, the pressure will be always continuous, but we lost the weak-compactness of $\mathcal{U}$. Consequently the existence of optimal controls fails. On the other hand, in practice, we cannot expect that the control variable is active all over the boundary. It may be active only on a part of the boundary and fixed on the other part. So we cannot expect the control variable is continuous all over the boundary.

Note that if the pressure $p(x)$ is continuous locally at each observation point, the point pressure observations still make sense. On the other hand, once a sensor is installed at the boundary point s^k, a control device (e.g., an actuator) cannot be installed at the same point. So we may assume that no controller is installed in a neighborhood of a sensor location. Let us denote $(L^\infty)^3 \cap (C^{0,\alpha}_{\text{loc}}(,\ ))^3$ the set of all functions in $(L^\infty)^3$ which is local in $(C^{0,\alpha}(,\ ))^3$ around each sensor location.

Theorem 2.2 *For each $\vec{u} \in (L^\infty)^3 \cap (C^{0,\alpha}_{loc}(,\ ))^3_{\perp M_0}$, the pressure $p(x, \vec{u})$ is locally $C^{0,\alpha}(,\ )$ at each sensor location.*

Proof. By (2.1) and Lemma 2.1 (c), we see that

$$(\frac{1}{2}I + \mathcal{K}^*)^{-1}(\vec{u}) \in (L^\infty)^3 \cap (C^{0,\alpha}_{\text{loc}}(,\ ))^3.$$

Since $\vec{n}$ is Lipschitz continuous, to prove the theorem, the only thing we have to show is that $\mathcal{S}_p :\ (L^\infty)^3 \cap (C^{0,\alpha}_{\text{loc}}(,\ ))^3 \to C^{0,\alpha}_s(,\ )$, the set of all functions that is $C^{0,\alpha}$ at each sensor location $s \in ,$. This is stated in the following lemma. ∎

Lemma 2.5 $\mathcal{S}_p: \ (L^\infty)^3 \cap (C^{0,\alpha}_{loc}(\ ,\ ))^3 \to C^{0,\alpha}_s(\ ,\ )$.

Proof. Let $f \in L^\infty(\ ,\ )$ be locally $C^{0,\alpha}(\ ,\ )$ in a neighborhood of each sensor location $s \in\ ,\ ,$ we need to show that the function

$$x \mapsto \int_\Gamma e_i(x,\xi) f(\xi)\, d\sigma_\xi$$

is locally $C^{0,\alpha}$ at s. Lemma 2.3 states that

$$x \mapsto \int_\Gamma e_i(x,\xi)\, d\sigma_\xi$$

is C^1 on $\ ,\ ,$ so we only need to prove that the function

$$x \mapsto T(x) = \int_\Gamma e_i(x,\xi)(f(\xi) - f(x))\, d\sigma_\xi$$

is locally $C^{0,\alpha}$ at s. Let $0 < \alpha < 1$ and $0 < d$ be such that

$$|f(x) - f(\xi)| \le C_1 |x - \xi|^\alpha, \quad \forall\, x, \xi \in\ ,\ , |s - x| \le d,\ |s - \xi| \le d.$$

Choose $0 < R < d$ such that

$$\langle \vec{n}(s), \vec{\xi} \rangle = 1 - \langle \vec{n}(s), \vec{n}(s) - \vec{\xi} \rangle \ge \frac{1}{2}, \quad \forall\, \xi \in\ ,\ , |s - \xi| \le R,$$

or

$$d\sigma_\xi = \frac{\rho d\rho d\theta}{\langle \vec{n}(s), \vec{\xi} \rangle} \le 2\rho d\rho d\theta, \quad \forall\, \xi \in\ ,\ , |s - \xi| \le R.$$

Now for $x \in\ ,$ with $r = 2|s - x| < R$, we try to show

$$|T(s) - T(x)| \le C|x - s|^\alpha,$$

where C is independent of x. Let $B(r) = \{\xi \in\ ,\ : |s - \xi| \le r\}$. The integral on $\ ,\ \setminus B(r)$ is clearly local $C^{0,\alpha}$ at s, thus the proof can be seen in the following inequalities.

$$\begin{aligned}
&\left| \int_{B(r)} \left[e_i(s,\xi)(f(\xi) - f(s)) - e_i(x,\xi)(f(\xi) - f(x)) \right] d\sigma_\xi \right| \\
\le\ & \int_{B(r)} C_1 |s - \xi|^{\alpha - 2}\, d\sigma_\xi + \int_{B(\frac{3}{2}r)} C_1 |s - \xi|^{\alpha - 2}\, d\sigma_\xi \\
\le\ & C_1 \int_0^{2\pi} \int_0^r \rho^{\alpha - 2} 2\rho\, d\rho\, d\theta + C_1 \int_0^{2\pi} \int_0^{\frac{3}{2}r} \rho^{\alpha - 2} 2\rho\, d\rho\, d\theta \\
\le\ & C' r^\alpha \\
=\ & C_2 |s - x|^\alpha.
\end{aligned}$$

Write

$$\begin{aligned} & e_i(s,\xi)(f(\xi)-f(s)) - e_i(x,\xi)(f(\xi)-f(x)) \\ = \; & [e_i(s,\xi)-e_i(x,\xi)]\,[f(\xi)-f(x)] - e_i(s,\xi)(f(s)-f(x)) \end{aligned}$$

and deal with each term.

$$\begin{aligned} & \left|\int_{\Gamma\setminus B(r)} e_i(s,\xi)\,d\sigma_\xi (f(s)-f(x))\right| \\ \leq \; & \left|\int_{\Gamma\setminus B(R)} e_i(s,\xi)\,d\sigma_\xi\right| |f(s)-f(x)| + \left|\int_{B(R)\setminus B(r)} e_i(s,\xi)\,d\sigma_\xi\right| |f(s)-f(x)| \\ \leq \; & C_1\left|\int_{\Gamma\setminus B(R)} e_i(s,\xi)\,d\sigma_\xi\right| |s-x|^\alpha + C_1'\int_0^{2\pi}\int_r^R \rho^{-2}2\rho\,d\rho\,d\theta|s-x| \\ \leq \; & C'|s-x|^\alpha + C"(\ln R - \ln r)|s-x| \\ \leq \; & C_3|s-x|^\alpha \end{aligned}$$

where we have used that

$$|\ln r|\cdot|s-x|^{1-\alpha}| = |\ln(2|s-x|)\cdot|s-x|^{1-\alpha}| \leq C'.$$

By (2.10),

$$|e_i(s,\xi)-e_i(x,\xi)| \leq M|s-\xi|^{-3}|s-x|.$$

Notice that $f\in L^\infty(,\,)$, $|s-x| = \frac{1}{2}r$ and $R>r$, there is $M'>0$ such that

$$|f(\xi)-f(x)| \leq M'|\xi-x|, \quad \text{a.e. } \xi\in ,\ \setminus B(R).$$

So

$$\begin{aligned} \left|\int_{\Gamma\setminus B(R)} [e_i(s,\xi)-e_i(x,\xi)][f(\xi)-f(x)]\,d\sigma_\xi\right| \; & \leq \; MM'\int_{\Gamma\setminus B(R)} |s-\xi|^{\alpha-2}\,d\sigma_\xi|s-x| \\ & \leq \; C_4|s-x|. \end{aligned}$$

Finally for $\xi\in B(R)\setminus B(r)$, we have $|\xi-x|\leq 2|\xi-s|$. We see that

$$\begin{aligned} & \left|\int_{B(R)\setminus B(r)} [e_i(s,\xi)-e_i(x,\xi)][f(\xi)-f(x)]\,d\sigma_\xi\right| \\ \leq \; & MC_1\int_{B(R)\setminus B(r)} |s-\xi|^{\alpha-3}|s-x|2^\alpha\,d\sigma_\xi \\ \leq \; & MC_1 2^\alpha\int_0^{2\pi}\int_0^R \rho^{\alpha-3}2\rho\,d\rho\,d\theta|s-x| \\ = \; & 4\pi MC_1\frac{2^\alpha}{1-\alpha}(R^{\alpha-1}-r^{\alpha-1})|s-x| \\ \leq \; & C_5|s-x|^\alpha. \end{aligned}$$

Combine all the above integrals and inequalities, we conclude that the function

$$x \to T(x) = \int_\Gamma e_i(x,\xi)(f(\xi) - f(x))\, d\sigma_\xi$$

is locally $C^{0,\alpha}$ at s. ▮

Let $g_k \in C^{0,\alpha}(, \cap B(s^k, d)), k = 1, 2, ..., m$, where $B(s^k, d) = \{\xi \in \mathbb{R}^3 : |s^k - \xi| \le d\}$. Let $Bu, Bl \in (L^\infty(,))^3$ such that $Bu(x) = Bl(x) = g_k(x),\ \forall\ x \in B(s^k, d)$. Define the feasible control set

$$\mathcal{U} = \{u \in (L^p(,))^3 : Bl(x) \le u(x) \le Bu(x),\ \forall\ x \in ,\ \}. \tag{2.11}$$

By Theorem 2.2, the boundary point pressure observations $p(s^k)$ in our LQR problem setting (1.4) make sense. Among all these pressure solutions, there is a unique solution p s.t.

$$\sum_{k=1}^m \mu_k |p(s^k) - Z_k|^2 = \min_{a\in\mathbb{R}} \sum_{k=1}^m \mu_k |p_0(s^k) + a - Z_k|^2. \tag{2.12}$$

A direct calculation yields that $p(x) = p_0(x) + a$ must satisfy

$$\sum_{k=1}^m \mu_k (p_0(s^k) + a - Z_k) = 0. \tag{2.13}$$

So the LQR problem (1.4) is well-posed.

3 Characterization of the Optimal Control

We establish an optimality condition of the LQR problem through a variational inequality problem (VIP). The characterization of the optimal control is then derived from the optimality condition.

Let $L^0(,)$ denote the vector space of all p.v.-integrable functions on , . For each $\vec{f} \in (L^0(,))^3$, we define the vector-valued truncation function

$$\left[\vec{f}\right]_{Bl}^{Bu} = \left\{[f_i(x)]_{Bl_i(x)}^{Bu_i(x)} = \begin{cases} Bu_i(x) & \text{if} \quad f_i(x) \ge Bu_i(x) \\ f_i(x) & \text{if} \quad Bl_i(x) < f_i(x) < Bu_i(x) \\ Bl_i(x) & \text{if} \quad f_i(x) \le Bl_i(x) \end{cases} \right\}.$$

The next theorem characterizes the solution to a variational inequality problem, which will be used to derive an optimality condition of our LQR problem.

Theorem 3.1 *[36] Let*

$$\mathcal{U} = \left\{ \vec{u} \in (L^p(,\ ))^n \mid \vec{B}l(x) \le \vec{u}(x) \le \vec{B}u(x),\ \text{a.e. } x \in\ ,\ , \vec{u} \perp M_0 \right\}.$$

Then for each $f \in (L^q(,\ ))^n$, u^f *is a solution to the variational inequality*

(VIP) $$\langle u^f - f, u - u^f \rangle \ge 0 \quad \forall\, u \in \mathcal{U}$$

if and only if

(3.1) $$u^f = [f + z^f]_{Bl}^{Bu}$$

where $z^f \in M_0$ *is defined in Theorem 3.2 by* $[f + z^f]_{Bl}^{Bu} \perp M_0$ *and* $\langle \cdot, \cdot \rangle$ *is the pairing on* $((L^q(,\ ))^n, (L^p(,\ ))^n)$.

Moreover, (3.1) is well-defined in the sense that if z^1 *and* z^2 *are two vectors in* M_0 *s.t.*

$$[f + z^1]_{Bl}^{Bu} \perp M_0 \quad \textit{and} \quad [f + z^2]_{Bl}^{Bu} \perp M_0,$$

then

(3.2) $$[f(x) + z^1(x)]_{Bl}^{Bu} = [f(x) + z^2(x)]_{Bl}^{Bu} \quad \text{a.e. } x \in\ ,\ .$$

In a Hilbert space setting, the above theorem is called a characterization of projection. When $\mathcal{U}$ is a convex closed subset of a Hilbert space H, for each $f \in H$, u_f is a solution to the VIP if and only if

$$u_f = P_{\mathcal{U}}(f),$$

i.e., u_f is the projection of f on $\mathcal{U}$. However, this result is not valid in general Banach spaces. Instead we prove a characterization of truncation, which is a special case of projection.

We need to show the existence of z^1 in (3.1). For later developments, we also need to show that there is $z^2 \in M_0$ such that

$$[f + z^f]_{Bl}^{Bu} \perp N.$$

The following existence theorem plays an important role in establishing the characterization of the optimal control and will be applied several times in the next two sections. This result generalizes Theorem 2.2 in [36] in two aspects due to our current requirements. The first is that a Lebesgue integrable function has to be replaced by a p.v.-integrable function. Consequently, some Lebesgue integral related tools can no longer be applied. The second is that two subspaces N_0 and M_0 are involved.

Theorem 3.2 *Let , be a bounded closed set in* $\mathbb{R}^n$ *and* $,_0 \subset ,$ *be a subset s.t. meas* $(,_1) > 0$ *where* $,_1 = , \setminus ,_0$. *Let* $\vec{g} \in (L^p(,_0))^n$ *and* $\vec{B}l, \vec{B}u \in (L^p(,))^n$ $(p \geq 2)$ *be given s.t.*

$$\vec{B}l(x) < -\vec{B} < \vec{B} < \vec{B}u(x) \quad \text{(a.e.)} \quad \forall\, x \in ,_1$$

where $\vec{B} = (B, \cdots, B)$ *is given by (3.9) and*

$$\vec{B}l(x) = \vec{g}(x) = \vec{B}u(x) \quad \forall\, x \in ,_0.$$

Assume that $N_0,\ M_0$ *are two* m*-dimensional subspace in* $(L^q(,))^n$ $(q \leq 2, \frac{1}{p} + \frac{1}{q} = 1)$ *and* $N_1 = \{\vec{z}|_{\Gamma_1} \mid \vec{z} \in N_0\}$, *then a necessary and sufficient condition that for each* $\vec{f} \in (L^0(,))^n$ *there exists* $\vec{z}_f \in M_0$ *s.t.*

$$\left[\vec{f}(x) + \vec{z}_f(x)\right]_{Bl}^{Bu} \perp N_0. \tag{3.3}$$

is that

$$\vec{g} \perp N_1^c = \{\vec{z}|_{\Gamma_0} \mid \vec{z} \in N_0, \vec{z}|_{\Gamma_1} = 0\}. \tag{3.4}$$

Proof. Let $z = (\vec{z}^1, \cdots, \vec{z}^m)$ be basis in M_0.
Case 1: $\dim(N_1) = \dim(N_0)$, i.e., $N_1^c = \{0\}$. Let $y = (\vec{y}^1, \cdots, \vec{y}^m)$ be an orthonormal basis in N_1 (in N_0 as well). To prove the first part of the theorem, we have to show that for each $\vec{f} \in (L^0(,))^n$, there exists $C^f = (c_1^f, \cdots, c_m^f) \in \mathbb{R}^m$ s.t.

$$\langle \left[\vec{f}(x) + \sum_{i=1}^m c_i^f \vec{z}^i(x)\right]_{Bl}^{Bu}, \vec{y}^j \rangle_\Gamma = 0, \quad \forall\, j = 1, \cdots, m.$$

For each $\vec{f} \in (L^0(,))^n$, we define a map $T_f : \mathbb{R}^m \to \mathbb{R}^m$, for $C = (c_1, \cdots, c_m) \in \mathbb{R}^m$, by

$$T_f(C) = \left\{ \langle \left[\vec{f}(x) + \sum_{i=1}^m c_i \vec{y}^i(x)\right]_{Bl}^{Bu}, \vec{z}^j \rangle_\Gamma \right\}_{j=1,\cdots,m}. \tag{3.5}$$

Then to prove the first part, it suffices to show that for each $\vec{f} \in (L^0(,))^n$, there exists $C_f \in \mathbb{R}^m$ s.t.

$$T_f(C_f) = 0.$$

It is easy to check that for any $\vec{f} \in (L^0(,))^n$ and $C_1, C_2 \in \mathbb{R}^m$, there exist a constant γ_1 independent of C_1 and C_2 s.t.

$$|T_f(C_1) - T_f(C_2)| \leq \gamma_1 |C_1 - C_2|. \tag{3.6}$$

So $C \to T_f(C)$ is a bounded (depends on Bl and Bu) Lipschitz continuous map. To show that T_f has a zero, we prove that there exists a constant $R > 0$ s.t. when $C \in \mathbb{R}^m$ and $|C| > R$, we have

$$T_f(C) \cdot C > 0. \tag{3.7}$$

Once (3.7) is verified, we have

$$\begin{aligned} |C - T_f(C)|^2 &= |C|^2 - 2T_f(C) \cdot C + |T_f(C)|^2 \\ &< |C|^2 + |T_f(C)|^2 \qquad \forall\, C \in \mathbb{R}^m, |C| > R. \end{aligned}$$

By Altman's fixed point theorem [20], the map $C \to C - T_f(C)$ has a fixed point $C^f \in B_R$ (B_R is the ball of radius R at the origin), i.e.,

$$T_f(C^f) = 0.$$

So it remains to verify (3.7). Define

$$D = \left\{ C = (c_1, \cdots, c_m) \in \mathbb{R}^m \mid \sum_{i=1}^m c_i^2 = 1 \right\}.$$

It suffices to show that there exists $R > 0$ s.t. for $t > R$,

$$T_f(tC) \cdot C > 0 \quad \forall C \in D.$$

In the following, we prove that for each given $\vec{f} \in (L^0(,))^n$ and $C \in D$, there exists $R > 0$ s.t. when $t > R$, we have

$$T_f(tC) \cdot C > 0 \quad \forall C \in D.$$

For each $C \in D$, we denote

$$\vec{y}^C(x) = \sum_{i=1}^m c_i \vec{y}^i(x) \quad \text{and} \quad \vec{z}^C(x) = \sum_{i=1}^m c_i \vec{z}^i(x).$$

It is obvious that

$$\int_{\Gamma_1} |\vec{y}^C(x)| d\sigma_x$$

is continuous in C and positive on the compact set D, hence

$$m_y = \min_{C \in D} \{ \int_{\Gamma_1} | \vec{y}^C(x)| d\sigma_x \} > 0 \tag{3.8}$$

and we set

$$B = \frac{\max_{C \in D} \int_{\Gamma_0} |\vec{g}(x) \cdot \vec{y}^C(x)| d\sigma_x}{m_y}. \tag{3.9}$$

For any given $\varepsilon > 0$, we assume

$$Bl_i(x) \le -B - \varepsilon, \quad Bu_i(x) \ge B + \varepsilon \quad \forall\, x \in \,,_1, \quad i = 1, \cdots, n.$$

For each $C \in D, t > 0$,

$$\begin{aligned} T_f(tC) \cdot C &= \sum_{j=1}^{m} \Big(\int_\Gamma \Big[\vec{f}(x) + \sum_{i=1}^{m} tc_i \vec{z}^i(x) \Big]_{Bl}^{Bu} \cdot \vec{y}^j(x) d\sigma_x \Big) c_j \\ &= \int_\Gamma \Big[\vec{f}(x) + t\vec{z}^C(x) \Big]_{Bl}^{Bu} \cdot \vec{y}^C(x) d\sigma_x \\ &= \int_{\Gamma_1} \Big[\vec{f}(x) + t\vec{z}^C(x) \Big]_{Bl}^{Bu} \cdot (\vec{y}^C(x)) d\sigma_x + \int_{\Gamma_0} \vec{g}(x) \cdot \vec{y}^C(x) d\sigma_x \\ &= \sum_{i=1}^{n} I_i^C(t) + \int_{\Gamma_0} \vec{g}(x) \cdot \vec{y}^C(x) d\sigma_x \end{aligned}$$

where for $i = 1, \cdots, n$,

$$I_i^C(t) = \int_\Gamma [f_i(x) + tz_i^C(x)]_{Bl_i(x)}^{Bu_i(x)} y_i^C(x) d\sigma_x.$$

Let

$$,_i^{C+} = \{x \in \,,_1 \mid y_i^C(x) > 0\} \quad \text{and} \quad ,_i^{C-} = \{x \in \,,_1 \mid y_i^C(x) < 0\}.$$

We have

$$\begin{aligned} \lim_{t \to +\infty} I_i^C &= \int_{\Gamma_i^{C+}} Bu_i(x) \cdot y_i^C(x) d\sigma_x + \int_{\Gamma_i^{C-}} Bl_i(x) \cdot y_i^C(x) d\sigma_x \\ &\ge (B + \varepsilon) \int_{\Gamma_1} |y_i^C(x)| d\sigma_x. \end{aligned}$$

Thus

$$\begin{aligned} \lim_{t \to +\infty} T_f(tC) \cdot C &\ge (B + \varepsilon) \sum_{i=1}^{n} \int_{\Gamma_1} |y_i^C(x)| d\sigma_x + \int_{\Gamma_0} \vec{g}(x) \cdot \vec{y}^C(x) d\sigma_x \\ &\ge (B + \varepsilon) \int_{\Gamma_1} |y^C(x)| d\sigma_x + \int_{\Gamma_0} \vec{g}(x) \cdot \vec{y}^C(x) d\sigma_x \\ &\ge \varepsilon m_y, \end{aligned}$$

where m_y given by (3.8) is independent of C. From (3.6), we see that $T_f(C) \cdot C$ is continuous in both $\vec{f}$ and C, therefore there exist $R^C > 0$ and δ_C, as $t > R^C$ and $|C' - C| < \delta_C$, we have

$$T_f(tC') \cdot C' \ge \frac{1}{2} \varepsilon m_y > 0.$$

Since D is compact, there exist $C_1, \cdots, C_s \in D$ and $\delta_1, \cdots, \delta_s$ s.t.

$$D \subset \cup_{k=1}^{s} B_{\delta_k}(C_k).$$

Let

$$R^0 = \max\{R^{C_1}, \cdots, R^{C_s}\} \quad \text{and} \quad r_0 = \min\{r^{C_1}, \cdots, r^{C_s}\}.$$

When $t > R^0$, we have

$$T_f(tC) \cdot C \geq \frac{1}{2}\varepsilon m_y > 0 \quad \forall\, C \in D.$$

So we only need to take

$$\vec{B} = (B, \cdots, B)$$

and

$$\vec{B}l < -\vec{B} < \vec{B} < \vec{B}u, \quad \text{a.e. on } ,\ _1.$$

Case 2: $m_1 = \dim(N_1) < \dim(N_0) = m$. Let $y = (\vec{y}^1, \cdots, \vec{y}^m)$ be an orthonormal basis in N_0, where $(\vec{y}^1, \cdots, \vec{y}^{m_1})$ is a basis in N_1 with

$$\vec{y}^i|_{\Gamma_0} = 0,\ (i = 1, \cdots, m_1) \quad \text{and} \quad \vec{y}^j|_{\Gamma_1} = 0,\ (j = m_1 + 1, \cdots, m). \tag{3.10}$$

By the proof in Case 1, for each $\vec{f} \in (L^1(,\ ))^n$, there exists $C^f = (c_1^f, \cdots, c_{m_1}^f) \in \mathbb{R}^{m_1}$ s.t.

$$\langle \Big[\vec{f}(x) + \sum_{i=1}^{m_1} c_i^f \vec{z}^i(x)\Big]_{Bl}^{Bu}, \vec{y}^j\rangle_{\Gamma_1} = 0, \quad \forall\ j = 1, \cdots, m_1.$$

Then for any $c_{m_1+1}^f, \cdots, c_m^f \in \mathbb{R}$, by (3.10), we have

$$\begin{aligned}\langle \Big[\vec{f}(x) + \sum_{i=1}^{m} c_i^f \vec{z}^i(x)\Big]_{Bl}^{Bu}, \vec{y}^j\rangle_{\Gamma} &= \langle \vec{g}(x), \vec{y}^j\rangle_{\Gamma_0} + \langle \Big[\vec{f}(x) + \sum_{i=1}^{m_1} c_i^f \vec{z}^i(x)\Big]_{Bl}^{Bu}, \vec{y}^j\rangle_{\Gamma_1} \\ &= 0, \quad \forall j = 1, \cdots, m_1.\end{aligned}$$

On the other hand, when $j > m_1$, for any $c_1, \cdots, c_m \in \mathbb{R}$, by (3.10), we have

$$\langle \Big[\vec{f}(x) + \sum_{i=1}^{m} c_i \vec{z}^i(x)\Big]_{Bl}^{Bu}, \vec{y}^j\rangle_{\Gamma} = \langle \vec{g}(x), \vec{y}^j\rangle_{\Gamma_0}.$$

Therefore

$$\langle \Big[\vec{f}(x) + \sum_{i=1}^{m} c_i \vec{z}^i(x)\Big]_{Bl}^{Bu}, \vec{y}^j\rangle_{\Gamma} = 0, \quad j > m_1,$$

if and only if

$$\langle \vec{g}(x), \vec{y}^j\rangle_{\Gamma_0} = 0, \quad j > m_1,$$

i.e., (3.4) is satisfied. The proof is complete. ∎

Remark 3.1 In the above theorem, , $_0$ can be regarded as the union of $\cup_{k=1}^{m}(, \cap B(s^k, d))$ and $g(x) = g_k(x)$ for $x \in$, $\cap B(s^k, d)$, $k = 1, 2, ..., m$. For simplicity, we may assume $g_k(x) \equiv 0$, so condition (3.4) is satisfied with $N_0 = N = \ker(\frac{1}{2}I + \mathcal{K}^*)$. When rigid body motion is considered,

$$M_0 = \{\vec{a} + \vec{b} \times \vec{x} \mid \vec{a}, \vec{b} \in \mathbb{R}^3\},$$

we have $\dim(M_0) = \dim(M_1) = \dim(N_0) = 6$, so all the conditions in the theorem are satisfied. Therefore for any $f \in (L^0(,))^3$ there exist $z_1^f, z_2^f \in M_0$ such that

$$[f + z_1^f]_{Bl}^{Bu} \perp M_0 \quad \text{and} \quad [f + z_2^f]_{Bl}^{Bu} \perp N.$$

On the other hand, if $Bl = -\infty$ and $Bu = +\infty$, there will be no bound constraints. so the above theorem deals with only a linear system, the conclusions clearly hold. ∎

Let $\langle \cdot, \cdot \rangle$ be the pairing on $((L^q(,))^3, (L^p(,))^3)$. Since our objective function $J(\vec{u})$ is strictly convex and differentiable, and the feasible control set $\mathcal{U}$ is a closed bounded convex subset in the reflexive Banach space $(L^p(,))^3$, the existence and uniqueness of the optimal control are well-established. It is then well-known that $\vec{u}^*$ is an optimal control of the LQR problem if and only if

(3.11) $$\langle \nabla J(\vec{u}^*), \vec{u} - \vec{u}^* \rangle \geq 0, \quad \forall\, \vec{u} \in \mathcal{U}.$$

For any $\alpha > 0$, (3.11) is equivalent to

(3.12) $$\langle \vec{u}^* - (\vec{u}^* - \alpha \nabla J(\vec{u}^*)), \vec{u} - \vec{u}^* \rangle \geq 0, \quad \forall\, \vec{u} \in \mathcal{U}.$$

Our characterization of truncation states that $\vec{u}^* \in \mathcal{U}$ is a solution to the VIP (3.12) if and only if

(3.13) $$\vec{u}^* = [\vec{u}^* - \alpha \nabla J(\vec{u}^*) + \vec{z}^*]_{Bl}^{Bu},$$

where $\vec{z}^* \in M_0$ is defined in Theorem 3.2 s.t.

$$[\vec{u}^* - \alpha \nabla J(\vec{u}^*) + \vec{z}^*]_{Bl}^{Bu} \perp M_0.$$

By Theorems 3.2 and 3.1, (3.13) is well-defined.

In optimal control theory it is important to obtain a state-feedback characterization of the optimal control, i.e., the optimal control is stated as an explicit function of the optimal state. So the optimal control can be determined by a physical measurement of the optimal state. Our efforts are devoted to derive such a result. It is clear from (3.13) that to derive a pressure state feedback characterization for the optimal control, we only need to find a formula for the gradient of the objective function, i.e., $\nabla J(\vec{u}^*)$. We need several lemmas.

Lemma 3.1 *For each fixed* $s \in ,$,

$$\mathcal{K} \circ \vec{e}(s, \cdot) \in (L^q(,))^3.$$

Proof. We only need to prove that

$$\left| \int_\Gamma \frac{(x_i - \xi_i)(x_j - \xi_j)(x - \xi) \cdot \vec{n}(\xi)}{|x - \xi|^5} \frac{s_j - \xi_j}{|s - \xi|^3} \, d\sigma_\xi \right| \le \frac{C(, , \alpha)}{|x - s|^{1+\alpha}} \tag{3.14}$$

for some $0 < \alpha < 1$.

$$\begin{aligned} & \frac{(x_i - \xi_i)(x_j - \xi_j)(x - \xi) \cdot \vec{n}(\xi)}{|x - \xi|^5} \\ = & \left[\frac{(x_i - \xi_i)(x_j - \xi_j)(x - \xi) \cdot \vec{n}(\xi)}{|x - \xi|^5} - \frac{(x_i - s_i)(x_j - s_j)(x - s) \cdot \vec{n}(s)}{|x - s|^5} \right] \\ & + \frac{(x_i - s_i)(x_j - s_j)(x - s) \cdot \vec{n}(s)}{|x - s|^5} \\ = & I_1 + I_2. \end{aligned}$$

The first term I_1 needs to be further decomposed. By adding and subtracting some terms, we have

$$\begin{aligned} I_1 = & \frac{(x - \xi) \cdot \vec{n}(\xi)}{|x - \xi|^5 |x - s|^2} \left[(x_i - \xi_i)(x_j - \xi_j)|x - s|^2 - (x_i - s_i)(x_j - s_j)|x - \xi|^2 \right] + \\ & \frac{(x_i - s_i)(x_j - s_j)}{|x - s|^2} \left[\frac{(x - s) \cdot \vec{n}(s)}{|x - s|^3} - \frac{(x - \xi) \cdot \vec{n}(\xi)}{|x - \xi|^3} \right] \\ = & I_{11} + I_{12}. \end{aligned}$$

By adding and subtracting some terms again, we can show that

$$I_{11} \le \frac{4|s - \xi|}{|x - \xi||x - s|}.$$

Applying inequality (2.5), we can see

$$I_{12} \le C'|s - \xi|^\alpha \left\{ \frac{1}{|x - \xi|^{1+\alpha}} + \frac{1}{|x - s|^{1+\alpha}} \right\}.$$

Then we have

$$\begin{aligned}
\left|\int_\Gamma I_{11}\frac{s_j-\xi_j}{|s-\xi|^3}\,d\sigma_\xi\right| &\le \frac{4}{|x-s|}\int_\Gamma \frac{1}{|x-s|}\frac{1}{|s-\xi|}\,d\sigma_\xi \quad (\text{ by (2.3)})\\
&\le \frac{4C'(\,,\,\alpha)}{|x-s|^{1+\alpha}},\\
\left|\int_\Gamma I_{12}\frac{s_j-\xi_j}{|s-\xi|^3}\,d\sigma_\xi\right| &\le C'\int_\Gamma\left\{\frac{1}{|x-\xi|^{1+\alpha}}\frac{1}{|s-\xi|^{2-\alpha}}+\frac{1}{|x-s|^{1+\alpha}}\frac{1}{|s-\xi|^{2-\alpha}}\right\}d\sigma_\xi\\
&\le \frac{C}{|x-s|^{1+\alpha}}
\end{aligned}$$

where to obtain the last inequality we have used (2.3); Since

$$|\langle\vec{n}(s),x-s\rangle|\le M|x-s|^2,$$

it is clear that

$$\left|\int_\Gamma I_2\,d\sigma_\xi\right|\le\frac{C'}{|x-s|}.$$

This conclues that the inequality (3.14) holds and the proof is complete. ∎

Lemma 3.2 *For $\beta_k, k=1,2,...,m$ with $\sum_{k=1}^m\beta_k=0$, we have $\sum_{k=1}^m\beta_k\vec{e}(s^k,\cdot)\perp N$ and $\mathcal{K}\circ(\sum_{k=1}^m\beta_k\vec{e}(s^k,\cdot))\perp N$.*

Proof. For any $\eta\in N$, we have

$$\mathcal{S}_p(\eta)=l_0=\int_\Gamma\vec{e}(x,\xi)\eta(\xi)\,d\sigma_\xi,\quad\forall x\in\,,\,.$$

So $\sum_{k=1}^m\beta_k\vec{e}(s^k,\cdot)\perp N$. To prove the second part, for each $\xi\in\,,\,$ set

$$\vec{e}_\varepsilon(s^k,\xi)=\begin{cases}\vec{e}(s^k,\xi), & |\xi-s^k|\ge\varepsilon,\\ 0, & |\xi-s^k|<\varepsilon.\end{cases}$$

Since $\sum_{k=1}^m\beta_k\vec{e}(s^k,\cdot)\perp N$,

$$\left|\int_\Gamma\sum_{k=1}^m\beta_k\vec{e}_\varepsilon(s^k,\xi)\eta(\xi)\,d\sigma_\xi\right|\le M\varepsilon,$$

for some constant M and for all $\varepsilon>0$. We see that

$$\left|\int_\Gamma\sum_{k=1}^m\beta_k\left\{\int_\Gamma Q(\xi,\zeta)\vec{e}(s^k,\zeta)\,d\sigma_\zeta\right\}\eta(\xi)\,d\sigma_\xi\right|$$

$$
\begin{aligned}
&= \left| \lim_{\varepsilon \to 0} \int_\Gamma \sum_{k=1}^m \beta_k \left\{ \int_\Gamma Q(\xi,\zeta) \vec{e}_\varepsilon(s^k,\zeta)\, d\sigma_\zeta \right\} \eta(\xi)\, d\sigma_\xi \right| \\
&= \left| \lim_{\varepsilon \to 0} \int_\Gamma \int_\Gamma \sum_{k=1}^m \beta_k Q(\xi,\zeta) \eta(\xi)\, d\sigma_\xi \vec{e}_\varepsilon(s^k,\zeta)\, d\sigma_\zeta \right| \\
&= \left| \lim_{\varepsilon \to 0} \int_\Gamma \sum_{k=1}^m \beta_k \mathcal{K}^*(\eta)(\zeta) \vec{e}_\varepsilon(s^k,\zeta)\, d\sigma_\zeta \right| \quad (\text{by } (\frac{1}{2}I + \mathcal{K}^*)(\eta)(\xi) = 0.) \\
&= \left| \lim_{\varepsilon \to 0} \int_\Gamma \sum_{k=1}^m \beta_k (-\frac{1}{2}\eta(\zeta)) \vec{e}_\varepsilon(s^k,\zeta)\, d\sigma_\zeta \right| \\
&\le \lim_{\varepsilon \to 0} M\varepsilon \\
&= 0.
\end{aligned}
$$

In the above, since the Fubini Theorem cannot be applied directly, a limit process has to be taken. ▮

With the above lemmas it is easy to show

Lemma 3.3 *For each* $h \in (L^p(\,,\,))^3_{\perp M_0}$ *with* $h(x) = 0, \forall x \in \,,\, _0 = \,,\, \cap (\cup_{k=1}^m B(s^k,d))$ *and* $\sum_{k=1}^m \beta_k = 0$, *we have*

$$
\begin{aligned}
&\langle \sum_{k=1}^m \beta_k \vec{e}(s^k,\cdot), (\frac{1}{2}I + \mathcal{K}^*)^{-1}(h)(\cdot) \rangle \\
&= \langle 2\sum_{k=1}^m \beta_k \vec{e}(s^k,\xi) - 2(\frac{1}{2}I + \mathcal{K})^{-1} \circ \mathcal{K}(\sum_{k=1}^m \beta_k \vec{e}(s^k,\cdot))(\xi), h(\xi) \rangle.
\end{aligned}
$$

Proof. Apply Lemmas 3.1 and 3.2, we see

$$
\begin{aligned}
&\langle \sum_{k=1}^m \beta_k \vec{e}(s^k,\cdot), (\frac{1}{2}I + \mathcal{K}^*)^{-1}(h)(\cdot) \rangle \\
&= \langle \sum_{k=1}^m \beta_k \vec{e}(s^k,\cdot), 2h(\cdot) - 2\mathcal{K}^* \circ (\frac{1}{2}I + \mathcal{K}^*)^{-1}(h)(\cdot) \rangle \\
&= \langle 2\sum_{k=1}^m \beta_k \vec{e}(s^k,\xi) - 2(\frac{1}{2}I + \mathcal{K})^{-1} \circ \mathcal{K}(\sum_{k=1}^m \beta_k \vec{e}(s^k,\cdot))(\xi), h(\xi) \rangle.
\end{aligned}
$$
▮

Now we are ready to present a pressure state feedback characterization of the optimal control.

Theorem 3.3 *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded domain with smooth boundary , . The LQR problem has a unique optimal control* $\vec{u}^* \in \mathcal{U}$ *and a unique optimal pressure state* p^* *which* *is $C^{0,\alpha}(,\ )$ locally at each observation point, s.t.*

$$\sum_{k=1}^{m}\beta_k \equiv \sum_{k=1}^{m}\mu_k(p^*(s^k)-Z_k)=0, \tag{3.15}$$

and $\forall x \in ,$

$$\begin{aligned}\vec{u}^*(x) &= \Bigg[\frac{1}{\gamma}\Bigg\{(\frac{1}{2}I+\mathcal{K})^{-1}\left(\sum_{k=1}^{m}\beta_k\vec{n}(s^k)T(s^k,\cdot)+\vec{a}_T+\vec{b}_T\times(\vec{\cdot})\right)(x)- \\ &\quad 2\sum_{k=1}^{m}\beta_k\vec{e}(s^k,x)-2(\frac{1}{2}I+\mathcal{K})^{-1}\circ\mathcal{K}\sum_{k=1}^{m}\beta_k\vec{e}(s^k,\cdot)(x)+\vec{a}+\vec{b}\times x\Bigg\}\Bigg]_{Bl}^{Bu},\end{aligned} \tag{3.16}$$

where $\vec{a}_T+\vec{b}_T\times(\vec{\cdot})$ and $\vec{a}+\vec{b}\times\vec{x}$ are in M_0 defined in Theorem 3.2 s.t.

$$\left(\sum_{k=1}^{m}\beta_k\vec{n}(s^k)T(s^k,\cdot)+\vec{a}_T+\vec{b}_T\times(\vec{\cdot})\right)\perp N \quad \textit{and} \quad \vec{u}^*\perp M_0$$

and M_0 is given in (1.2).

Proof. Let X be the space of all functions $h \in (L^p(,\ ))^3_{\perp M_0}$ with $h(x)=0, \forall x \in\ ,\ _0 =\ ,\ \cap(\cup_{k=1}^m B(s^k,d))$. Since our objective function $J(\vec{u})$ is strictly convex and differentiable, and the feasible control set $\mathcal{U}$ is a closed bounded convex subset in the reflexive Banach space X, the existence and uniqueness of the optimal control are well-established. Equation (3.15) is just a copy of (2.13). By our characterization of truncation, Theorem 3.1 with $\alpha=\frac{1}{2\gamma}$,

$$\vec{u}^*(x)=\left[\vec{u}^*(x)-\frac{1}{2\gamma}\nabla J(\vec{u})(x)+\vec{a}+\vec{b}\times\vec{x}\right]_{Bl}^{Bu}, \quad \forall\, x\in\ ,$$

where $\vec{a}+\vec{b}\times\vec{x}\in M_0$ is defined in Theorem 3.2 s.t.

$$\left[\vec{u}^*-\frac{1}{2\gamma}\nabla J(\vec{u})+\vec{a}+\vec{b}\times\vec{x}\right]_{Bl}^{Bu}\perp M_0.$$

To prove (3.16), we only need to show that for each $\xi\in\ ,\ \setminus\ ,\ _0$,

$$\begin{aligned}\nabla J(u^*)(\xi) &= -2\Bigg\{(\frac{1}{2}I+\mathcal{K})^{-1}\left(\sum_{k=1}^{m}\beta_k\vec{n}(s^k)T(s^k,\cdot)+\vec{a}_T+\vec{b}_T\times(\vec{\cdot})\right)(\xi)- \\ &\quad 2\sum_{k=1}^{m}\beta_k\vec{e}(s^k,\xi)+2(\frac{1}{2}I+\mathcal{K})^{-1}\circ\mathcal{K}\sum_{k=1}^{m}\beta_k\vec{e}(s^k,\cdot)(\xi)+\vec{a}+\vec{b}\times\xi\Bigg\}+2\gamma u(\xi).\end{aligned} \tag{3.17}$$

Since $\nabla J(\vec{u})$ defines a bounded linear functional on X, for any $\vec{h} \in X$, take (1.14) into account, we have

$$
\begin{aligned}
& \langle \nabla J(\vec{u}), \vec{h} \rangle \\
= \; & 2\sum_{k=1}^{m} \beta_k \frac{1}{2}\vec{n}(s^k)(\frac{1}{2}I + \mathcal{K}^*)^{-1}(\vec{h})(s^k) + 2\sum_{k=1}^{m} \beta_k \mathcal{S}_p(\frac{1}{2}I + \mathcal{K}^*)^{-1}(\vec{h})(s^k) + 2\gamma\langle \vec{u}, \vec{h} \rangle \\
= \; & I_1 + I_2 + I_3,
\end{aligned}
$$

where

$$
I_3 = 2\gamma\langle \vec{u}, \vec{h} \rangle,
$$

$$
\begin{aligned}
I_2 &= 2\sum_{k=1}^{m} \beta_k \mathcal{S}_p((\frac{1}{2}I + \mathcal{K}^*)^{-1}\vec{h})(s^k) \\
&= 2\int_\Gamma \sum_{k=1}^{m} \beta_k \vec{e}(s^k, \cdot), (\frac{1}{2}I + \mathcal{K}^*)^{-1}(\vec{h})(\cdot)\, d\sigma_\xi \quad \text{(by Lemma 3.3)} \\
&= 2\langle 2\sum_{k=1}^{m} \beta_k \vec{e}(s^k, \xi) - 2(\frac{1}{2}I + \mathcal{K})^{-1} \circ \mathcal{K} \sum_{k=1}^{m} \beta_k \vec{e}(s^k, \cdot)(\xi) + \vec{a} + \vec{b} \times \xi, \vec{h}(\xi) \rangle
\end{aligned}
$$

and

$$
\begin{aligned}
I_1 &= 2\sum_{k=1}^{m} \beta_k \frac{1}{2}\vec{n}(s^k)(\frac{1}{2}I + \mathcal{K}^*)^{-1}(\vec{h})(s^k) \\
&= \sum_{k=1}^{m} \beta_k \vec{n}(s^k)(-2\mathcal{K}^*(\frac{1}{2}I + \mathcal{K}^*)^{-1}(\vec{h})(s^k)) \\
&= -2\sum_{k=1}^{m} \beta_k \vec{n}(s^k) \int_\Gamma T(s^k, \xi)(\frac{1}{2}I + \mathcal{K}^*)^{-1}(\vec{h})(\xi)\, d\sigma_\xi \\
&= -2\int_\Gamma \left\{ \sum_{k=1}^{m} \beta_k \vec{n}(s^k) T(s^k, \xi) + \vec{a}_T + \vec{b}_T \times (\vec{\cdot}) \right\} (\frac{1}{2}I + \mathcal{K}^*)^{-1}(\vec{h})(\xi)\, d\sigma_\xi \\
&= -2\int_\Gamma (\frac{1}{2}I + \mathcal{K})^{-1} \left(\sum_{k=1}^{m} \beta_k \vec{n}(s^k) T(s^k, \cdot) + \vec{a}_T + \vec{b}_T \times (\vec{\cdot}) \right) (\xi) \vec{h}(\xi)\, d\sigma_\xi
\end{aligned}
$$

where $\vec{a}_T + \vec{b}_T \times (\vec{\cdot}) \in M_0$ is given by Theorem 3.2 and Remark 3.1 such that

$$
\sum_{k=1}^{m} \beta_k \vec{n}(s^k) T(s^k, \cdot) + \vec{a}_T + \vec{b}_T \times (\vec{\cdot}) \perp N.
$$

Finally for each $\xi \in$, $\setminus$, $_0$, we have

$$
\nabla J(u^*)(\xi)
$$

$$= \quad -2(\frac{1}{2}I+\mathcal{K})^{-1}\left(\sum_{k=1}^{m}\beta_k\vec{n}(s^k)T(s^k,\cdot)+\vec{a}_T+\vec{b}_T\times(\vec{\cdot})\right)(\xi)+$$

$$2\left\{2\sum_{k=1}^{m}\beta_k\vec{e}(s^k,\xi)-2(\frac{1}{2}I+\mathcal{K})^{-1}\circ\mathcal{K}\sum_{k=1}^{m}\beta_k\vec{e}(s^k,\cdot)(\xi)+\vec{a}+\vec{b}\times\xi\right\}+2\gamma u(\xi).$$

So (3.17) is verified and the proof is complete. ∎

4 Robust Control Problems

In this section, we consider robust boundary control problems with boundary point velocity observations, i.e., to find a Neumann boundary control such that the velocity state is as close as possible to given target values at several observation points while avoiding a build-up of the pressure state at those observation points.

Consider to find $\vec{u}(x)\in\mathcal{U}$ such that $\vec{w}(x)$ and $p(x)$ satisfy (1.1) and so as to

$$\text{(4.18)} \qquad \min J(\vec{u})=\sum_{k=1}^{m}\mu_k|\vec{w}(s^k)-\vec{Z}_k^0|^2+\sum_{k=1}^{m}\nu_k|p(s^k)-Z_k^1|^2+\gamma\int_{\Gamma_1}|\vec{u}(x)|^2d\sigma_x.$$

The above is just a combination of LQR problems (1.3) and (1.4), by the results in [36] and in the last section, if we write

$$\vec{\alpha}_k=\mu_k(\vec{w}^*(s^k)-\vec{Z}_k) \quad \text{and} \quad \beta_k=\nu_k(p^*(s^k)-Z_k),$$

then the optimal control is given by

$$\text{(4.19)} \qquad \sum_{k=1}^{m}\vec{\alpha}_k=0,\quad \sum_{k=1}^{m}\vec{\alpha}_k\times\vec{s^k}=0 \quad\text{and}\quad \sum_{k=1}^{m}\beta_k=0,$$

and $\forall x\in$,

$$\vec{u}^*(x) \quad = \quad \left[\frac{1}{\gamma}\left\{-(\frac{1}{2}I+\mathcal{K})^{-1}\Big(\sum_{k=1}^{m}E(s^k,\cdot)\vec{\alpha}_k+\sum_{k=1}^{m}\beta_k\vec{n}(s^k)T(s^k,\cdot)+\vec{a}_T+\vec{b}_T\times(\vec{\cdot})\Big)(x)-\right.\right.$$

$$\text{(4.20)} \qquad \left.\left.2\sum_{k=1}^{m}\beta_k\vec{e}(s^k,x)-2(\frac{1}{2}I+\mathcal{K})^{-1}\circ\mathcal{K}\sum_{k=1}^{m_1}\beta_k\vec{e}(s^k,\cdot)(x)+\vec{a}+\vec{b}\times x\right\}\right]_{Bl}^{Bu},$$

where $\vec{a}_T+\vec{b}_T\times(\vec{\cdot})$ and $\vec{a}+\vec{b}\times\vec{x}$ are in M_0 defined in Theorem 3.2 and Remark 3.1 s.t.

$$\left(\sum_{k=1}^{m}\beta_k\vec{n}(s^k)T(s^k,\cdot)+\vec{a}_T+\vec{b}_T\times(\vec{\cdot})\right)\perp N \quad\text{and}\quad \vec{u}^*\perp M_0.$$

It is clear that the term to the right hand side (4.20) before the truncation by Bl and Bu is continuous everywhere on , except at $s^k, k = 1, 2, .., m$. While in each $B(s^k, d) \cap$, , $\vec{u} \equiv \vec{g}^k$, a given $C^{0,\alpha}$ continuous function, so if Bl and Bu are continuous in , $\setminus (\cup B(s^k, d))$, then $\vec{u}$ is continuous everywhere on , except on the boundary $\partial B(s^k, d) \cap$, . Notice that when only point velocity observations on the boundary are utilized in the system [36], we do not need the condition that $\vec{u}(x) \equiv \vec{g}^k(x)\ \forall x \in B(s^k, d) \cap$, , which means that the control is not active there, this condition is required due to point pressure observations on the boundary. It is then quite natural to think that one may be able to get rid of or relax this extra condition by using point velocity observations and distributed pressure observations around each observation point on the boundary. Notice that the point here is to bring the pressure down around each observation point, it does not matter whether we use point or distributed pressure observations. Our analysis shows that when distributed pressure observations are used to replace point pressure observations, this condition is still required to avoid oscillation of the optimal control and to reduce high peak of the pressure around each observation point.

Consider the following robust optimal control problem with point velocity observations and distributed pressure observations on the boundary, i.e., to find $\vec{u}(x) \in \mathcal{U}$ such that $(\vec{w})(x)$ and $p(x)$ satisfy (1.1) and so as to

$$\min J(\vec{u}) = \sum_{k=1}^{m} \mu_k |\vec{w}(s^k) - \vec{Z}_k|^2 + \int_{\Gamma_0} |p(\xi) - Z(\xi)|^2 \, d\sigma_\xi + \gamma \int_{\Gamma_1} |\vec{u}(x)|^2 d\sigma_x, \tag{4.21}$$

where $\Gamma_0 = \cup_{k=1}^{m}$(, $\cap\, B(s^k, d))$ for some $d > 0$. By analysis similar to deriving (3.15) and Lemma 3.2, for the optimal control $\vec{u}^*$ we see

$$\int_{\Gamma_0} (p(\xi, \vec{u}^*) - Z(\xi)) \, d\sigma_\xi = 0,$$

$$\int_{\Gamma_0} (p(\xi, \vec{u}^*) - Z(\xi)) \vec{e}(\xi, \zeta) \, d\sigma_\xi \perp N$$

and

$$\mathcal{K} \circ \Big(\int_{\Gamma_0} (p(\xi, \vec{u}^*) - Z(\xi)) \vec{e}(\xi, \cdot) \, d\sigma_\xi \Big)(\zeta) \perp N.$$

Then by applying (3.13) and deriving ∇J, we can show that for each $x \in$, ,

$$\vec{u}^*(x) \;\; = \;\; \Big[-\frac{1}{\gamma} \Big(2 \sum_{k=1}^{m} \mu_k (\vec{w}(s^k) - \vec{Z}_k) \cdot E(s^k, x) + 2 \int_{\Gamma_0} (p(\zeta) - Z(\zeta)) \vec{e}(\zeta, x) \, d\sigma_\zeta$$

$$-2(\frac{1}{2}I+\mathcal{K})^{-1}\circ\mathcal{K}\Big(\sum_{k=1}^{m}\mu_k(\vec{w}(s^k)\vec{Z}_k)\cdot E(s^k,\cdot)+\int_{\Gamma_0}(p(\zeta)-Z(\zeta))\vec{e}(\zeta,\cdot)\,d\sigma_\zeta\Big)(x)$$

$$+\vec{a}+\vec{b}\times x\Big)\Big]_{Bl}^{Bu}, \tag{4.22}$$

where before the truncation by Bu and Bl, the first term is in $(L^q(,\ ))^3$, the second in $(L^p(,\ ))^3$ and others are in $(C^{0,\alpha}(,\ )^3$. So when $x\to s^k$ the first term

$$2\sum_{k=1}^{m}\mu_k(\vec{w}(s^k)-\vec{Z}_k)\cdot E(s^k,x)$$

oscillates between $-\infty$ and $+\infty$ (see [36]) and dominates all the other terms. Consequently around each observation point, $\vec{u}$ oscillates between the bound Bu and Bl and always touches the bound. Such an optimal control will be very hard to realize unless $Bl=Bu=\vec{g}$ around each observation point. On the other hand, we have

$$\begin{aligned} p(x) &= \frac{1}{2}\vec{n}(x)\cdot(\frac{1}{2}I+\mathcal{K}^*)^{-1}(\vec{u})(x)+\mathcal{S}_p\circ(\frac{1}{2}I+\mathcal{K}^*)^{-1}(\vec{u})(x)\\ &= \vec{n}(x)\cdot\vec{u}(x)-2\mathcal{S}_p(\vec{u})(x)\\ &\quad -\vec{n}(x)\cdot(\frac{1}{2}I+\mathcal{K}^*)^{-1}\circ\mathcal{K}^*(\vec{u})(x)-2\mathcal{S}_p\circ(\frac{1}{2}I+\mathcal{K}^*)^{-1}\circ\mathcal{K}^*(\vec{u})(x)\end{aligned}$$

where the first two terms are in $(L^p(,\ ))^3$ and others in $(C^{0,\alpha}(,\ )^3$. So the pressure p also oscillates around each observation point which implies that the pressure has a peak around each observation point unless the difference $(Bu-Bl)$ is small, which means that the control is less active, around each observation point.

Finally we point out that our approach can be extended to study optimal nonlinear boundary control problems of the Stokes system and our characterization of variational inequality, Theorem 3.1 and (3.13) can be used to design a fast and uniform convergent numerical method (conditioned gradient projection method) to approximate the optimal control.

References

[1] H. T. Banks and K. Ito, *Structural actuator control of fluid/structure interactions*, in the Proceedings of the 33rd Conference on Decision and Control, Florida, 1994, pp. 283-288.

[2] J. A. Burns and Y. Ou, *Feedback control of the driven cavity problem using LQR designs*, in the Proceedings of the 33rd Conference on Decision and Control, Florida, 1994, pp. 289-294.

[3] J. A. Burns, "Optimal Control Methods for Inverse Design in Aerospace Systems", IFIP WG 7.2 Conference on Optimal Control: Methods, Algorithms, Applications, Gainesville, FL., Feb., 1997.

[4] A. E. Bryson, Jr. and Y.C. Ho, Applied Optimal Control, Hemisphere Publishing Corporation, 1975.

[5] E. Casas, *Control of an elliptic problem with pointwise state constraints*, SIAM J. Control Optim., 24(1986), pp. 1309-1318.

[6] E. Casas, *Boundary control of semilinear elliptic equations with pointwise state constraints*, SIAM J. Control Optim., 31(1993), pp. 993-1006.

[7] G.Chen and J.Zhou, Boundary Element Methods, Academic Press, London-San Diego, 1992.

[8] M. Christ, *Lectures on Singular Integral Operators*, AMS Regional Conference Series on Math., No 77, 1990.

[9] R. Coifman, A. McIntoch and I. Mayer, *L'intégral de Cauchy definit un operateur borné sur L^2 pour les curbs Lipschitziennes*, Ann. of Math., 116(1982), pp. 361-387.

[10] C. Cuvelier, *Optimal control of a system governed by the Navier-Stokes equations coupled with heat equation*, in "New Developments in Differential Equations", ed. by W. Eckhaus, North-Holland, 1976.

[11] E. J. Dahlberg, C. E. Hening and G. C. Verchota, *Boundary value problems for the systems of elastostatics in Lipschitz domains*, Duke Math. J., 57(1988), pp. 795-818.

[12] V.F. Demyanov and A.M. Rubinov, *Approximate methods in optimization problems*, American Elsevier, New York, 1971.

[13] Z. Ding, Topics on Potential Theory on Lipschitz Domains and Boundary Control Problems, Ph.D. Dissertation, Texas A& M University, College Station, TX, August, 1994.

[14] Z.Ding, L. Ji and J.Zhou, *Constrained LQR problems in elliptic distributed control systems with point observations*, SIAM J. Control Optim., 34(1996), pp.264-294.

[15] Z.Ding and J.Zhou, *Constrained LQR problems governed by the potential equation on Lipschitz domain with point observations*, J. de Mathematiques Pures et Appliquées, 74(1995), pp. 317-344.

[16] Z. Ding and J. Zhou, *Constrained LQR problems in elliptic distributed control systems with point observations – convergence results*, Appl. Math. Optim., to appear.

[17] Z. Ding, *Regularity of hydrostatic potentials and boundary control of the linearized Stokes system*, Preprint.

[18] E.B. Fabes, M. Jodeid, Jr., and N. M. Riviere, *Potential techniques for boundary value problems on C^1 domains*, Acta Math., 141(1978), pp. 165-186.

[19] E.B. Fabes, C. E. Henig and G. C. Verchota, The Dirichlet problems for the Stokes system on Lipschitz domains, Duke Math. J., 57(1988), pp. 769-793.

[20] J. Dugundji and A. Granas, *Fixed Point Theory*, PWN-Polish Scientific Publishers, Warszawa, 1982.

[21] V. Girault and P. A. Raviart, *Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms*, Springer, New York, 1986.

[22] M. Gunzburger, L. Hou and T. Svobodny, *Boundary velocity control of incompressible flow with an application to viscous drag reduction*, SIAM J. Con. Opt., 30(1992), pp. 167-181.

[23] K. Ito and S. Kang, *A dissipative feedback control synthesis for systems arising in fluid dynamics*, SIAM J. Contr. Optim., Vol. 32(1994), pp. 831-854.

[24] L.Ji and G.Chen, *Point observation in linear quadratic elliptic distributed control systems*, in Proceedings of AMS Summer Conference on Control and Identification of Partial Differential Equations, SIAM, Philadelphia, (1993), pp. 155-170.

[25] R. Kress, *Linear Integral Equations*, Springer-Verlag, New York, 1989.

[26] O.A. Ladyzhenskaia, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach Science Publishers, New Yor, 1963.

[27] J.L.Lions, *Contrôle optimal des systèms gouvernés par deséquations aux dèrivèes partielles*, Dunod, Gauthier-Villars, Paris, 1968.

[28] C.E. Lee and H.E. Taylor, *Fiber-optic Fabry-Perot temperature sensor using a low-coherence light source*, J. Lightwave technol., Vol. 9, no. 1, (1994), pp.129-134.

[29] V. G. Maz'ya and S. M. Nikol'skij, *Analysis IV: Linear and Boundary Integral Equations*, Encyclopaedia of Math. Science., Vol. 27, Springer, New York, 1991.

[30] R. Temam, *Navier-Stokes Equations*, North-Holland, New York, 1977.

[31] R. Temam, *Navier-Stokes Equations and Nonlinear Functional Analysis*, SIAM, Philadelphia, 1983.

[32] G.Verchota, *Layer potentials and boundary value problems for Laplace's equation on Lipschitz domains*, Ph.D thesis, University of Minnesota, 1982.

[33] M.C. Wang, *Fiber-optic Fabry-Perot temperature and dynamic sensor system using a low-coherence LED light source*, Ph.D. dissertation, Texas A&M University, College Station, Texas, May, 1995.

[34] M.T. Wlodarczyk and G. He, *A fiber-optic combustion pressure sensor system for automotive engine control*, Sensors, Vol. 11, no. 6, (1994), pp. 35-42.

[35] P. You and J. Zhou, *Constrained LQR problems in elliptic distributed control systems with point observations—on convergence rates*, SIAM J. Contr. Optim., to appear.

[36] P. You, Z. Ding and J. Zhou, *Optimal boundary control of Stokes' fluids with point velocity observations*, SIAM J. Contr. Optim., to appear.

[37] P. You, *Analysis and Computation of Nonlinear Boundary Control for Heat Conduction in Nonsmooth Domain with Point Observations*, Ph.D. dissertation, Texas A&M University, College Station, Texas, August, 1996.

[38] P. You and J. Zhou, *The heat equation with nonlinear boundary conditions in nonsmooth domain: a boundary integral equation approach with point observations*, in review.