Local Characterizations of Saddle Points and Their Morse Indices

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In honor of Professor David Russell on the occasion of his 60-birthday

Abstract

In this paper, numerically computable bound estimates of Morse indices of saddle points are established through their new local minimax type characterizations. The results provide methods for instability analysis in system design and control theory.

1 Introduction

Let H be a Hilbert space and $J : H \to \mathbb{R}$ be a Frechet differentiable functional. Denote by J' its Frechet derivative and J'' its second Frechet derivative if it exists. A point $\hat{u} \in H$ is a *critical point* of J if

$$J'(\hat{u}) = 0$$

as an operator $J' : H \to H$. A number $c \in \mathbb{R}$ is called a *critical value* of J if $J(\hat{u}) = c$ for some critical point $\hat{u}$. For a critical value c, the set $J^{-1}(c)$ is called a *critical level*. When the second Frechet derivative J'' exists at a critical point $\hat{u}$, $\hat{u}$ is said to be *non-degenerate* if $J''(\hat{u})$ is invertible as a linear operator $J''(\hat{u}) : H \to H$, otherwise $\hat{u}$ is said to be *degenerate*. The first candidates for a critical point are the local maxima and minima to which the classical critical point theory was devoted in calculus of variation. Traditional numerical methods focus on finding such stable solutions. Critical points that are not local extrema are *unstable*

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and called *saddle points*, that is, critical points u^* of J, for which any neighborhood of u^* in H contains points v, w s.t. $J(v) < J(u^*) < J(w)$. In physical systems, saddle points appear as unstable equilibria or transient excited states.

According to the Morse theory, the *Morse Index* (MI) of a critical point $\hat{u}$ of a real-valued functional J is the maximal dimension of a subspace of H on which the operator $J''(\hat{u})$ is negative definite; the *nullity* of a critical point $\hat{u}$ is the dimension of the null-space of $J''(\hat{u})$. So for a non-degenerate critical point, if its MI $= 0$, then it is a local minimizer and a stable solution, and if its $MI > 0$, then it is a min-max type and unstable solution.

Multiple solutions with different performance and instability exist in many nonlinear problems in natural and social sciences [21, 18, 13, 22, 12]. Stability is one of the main concerns in system design and control theory. Morse index of a solution can be used to measure its instability for some variational problems [4,20]. However in many applications, performance or maneuverability is more desirable, in particular, in system design or control of emergency or combat machineries. Usually instable solutions have much higher maneuverability or performance indices. For providing choice or balance between stability and maneuverability or performance, it is important to solve for multiple solutions and their Morse indices.

When cases are variational, they become multiple critical point problems. So it is important for both theory and applications to numerically solve for multiple critical points and their Morse indices in a stable way. So far, little is known in the literature to devise such a feasible numerical algorithm. Minimax principle is one of the most popular approaches in critical point theory. However, most minimax theorems in the literature (See [1], [15], [16], [18], [22]), such as the mountain pass, various linking and saddle point theorems, require one to solve a two-level *global* optimization problem and therefore not for algorithm implementation.

In [10], motivated by the numerical works of Choi-McKenna (6) and Ding-Costa-Chen (8), the Morse theory and the idea to define a solution submanifold, new local minimax theorems which characterize a critical point as a solution to a two-level *local* optimization problem are established. Based on the local characterization, a new numerical minimax method for finding multiple critical points is devised. The numerical method is implemented successfully to solve a class of semilinear elliptic PDE on various domains for multiple solutions [10]. Although Morse index has been printed for each numerical solution obtained in [10], their mathematical verifications have not been established. In [2], by using a global minimax principle, A. Bahri and P.L. Lions established some lower bound estimates for Morse indices of solutions to a class of semilinear elliptic PDE. There are also some efforts in the literature to numerically compute the Morse index of a solution to a class of semilinear elliptic PDE. In addition to finding a saddle point, one has to solve a corresponding linearized elliptic PDE at the solution for its first a few eigen-values. It is very expensive.

Since Morse index reflects *local* structure of a critical point, in this paper we show that our local minimax characterization enables us to establish more precise estimates for the Morse index of a saddle point in a more general setting. By our results the Morse index of a saddle point based on the local minimax method can be estimated even before we numerically compute the saddle point. So no extra work is required in addition to computation for the saddle point. In the last section, new local characterization of saddle points which are more general than minimax solutions and bound estimates for their Morse indices will be developed. In the rest of this section, we introduce some notations and theorems from [10] for future use.

For any subspace $H' \subset H$, let $S_{H'} = \{v | v \in H', \|v\| = 1\}$ be the unit sphere in H'. Let L be a closed subspace in H, called a base space, and $H = L \bigoplus L^\perp$ be the orthogonal decomposition where $L^\perp$ is the orthogonal complement of L in H. For each $v \in S_{L^\perp}$, we define a closed half space $[L, v] = \{tv + sw | w \in L,\ t \geq 0, s \in \mathbb{R}\}$.

Definition 1.1 *A set-valued mapping* $P\colon S_{L^\perp} \to 2^H$ *is called the* peak mapping *of* J *w.r.t.* $H = L \bigoplus L^\perp$ *if for any* $v \in S_{L^\perp}$, $P(v)$ *is the set of all local maximum points of* J *on* $[L, v]$. *A single-valued mapping* $p\colon S_{L^\perp} \to H$ *is a* peak selection *of* J *w.r.t.* L *if*

$$p(v) \in P(v) \quad \forall v \in S_{L^\perp}.$$

For a point $v \in S_{L^\perp}$, *we say* J *has a* local peak selection *w.r.t.* L *at* v, *if there is a neighborhood* $\mathcal{N}(v)$ *of* v *and a mapping* $p\colon \mathcal{N}(v) \cap S_{L^\perp} \to H$ *such that*

$$p(u) \in P(u) \quad \forall u \in \mathcal{N}(v) \cap S_{L^\perp}.$$

Most minimax theorems in critical point theory require one to solve a two-level global minimax problem and not for algorithm implementation. Our local minimax algorithm requires one to solve only unconstrained local maximizations at the first level. As pointed in [10], numerically it is great. However, theoretically, it causes three major problems: (a) for some $v \in S_{L^\perp}$, $P(v)$ may contain multiple local maxima in $[L, v]$. In particular, P may contain multiple branches, even U-turn or bifurcation points; (b) p may not be defined at some points in $S_{L^\perp}$; (c) the limit of a sequence of local maximum points may not be a local maximum point. So the analysis involved becomes much more complicated. We have been devoting great efforts to solve these three problems. We solve (a) and (b) by using a local peak selection. Numerically it is done by following certain negative gradient flow and developing some consistent strategies to avoid jumps between different branches of P. As for Problem (c), numerically we showed in [11] that as long as a sequence generated by the algorithm converges, the limit yields a saddle point. New local characterization of saddle points in this paper will further help us to solve those problems.

The following two local characterizations of saddle points are established in [10] and played important role in our local theory. We then provide some bound estimates of Morse indices of solutions based upon these two local characterizations.

Lemma 1.1 *Let $v_\delta \in S_{L^\perp}$ be a point. If J has a local peak selection p w.r.t. L at v_δ such that p is continuous at v_δ and $dis(p(v_\delta), L) > \alpha > 0$ for some $\alpha > 0$, then either $J'(p(v_\delta)) = 0$ or for any $\delta > 0$ with $\|J'(p(v_\delta))\| > \delta$, there exists $s_0 > 0$, such that*

$$J(p(v(s))) - J(p(v_\delta)) < -\alpha\delta\|v(s) - v_\delta\|$$

for any $0 < s < s_0$ and

$$v(s) = \frac{v_\delta + sd}{\|v_\delta + sd\|}, \quad d = -J'(p(v_\delta)).$$

The above result indicates that $v(s)$ defined in the lemma represents a direction for certain negative gradient flow of $J(p(\cdot))$ from v. So it is clear that if $p(v_0)$ is a local minimum point of J on any subset containing the path $p(v_0(s))$ for some small $s > 0$ then $J'(p(v_0)) = 0$. In particular, when we define a solution manifold

$$\mathcal{M} = \Big\{p(v) : v \in S_{L^\perp}\Big\},$$

we have $p(v(s)) \subset \mathcal{M}$. A solution submanifold was first introduced by Nehari in a study of a dynamic system [14] and then by Ding-Ni in study of semilinear elliptic PDE [16] and they prove that a global minimum point of a generic energy function J on the solution submanifold $\mathcal{M}$ w.r.t. $L = \{0\}$ is a saddle point basically with MI= 1. It is clear that our definition generalizes the notion of solution (stable) submanifold.

Theorem 1.1 *Let $v_0 \in S_{L^\perp}$ be a point. If J has a local peak selection p w.r.t. L at v_0 s.t.*

(i) *p is continuous at v_0,*

(ii) *$dis(p(v_0), L) > 0$ and*

(iii) *v_0 is a local minimum point of $J(p(v))$ on $S_{L^\perp}$.*

Then $p(v_0)$ is a critical point of J.

The following PS condition will be used to replace the usual compact condition.

Definition 1.2 *A function $J \in C^1(H)$ is said to satisfy the Palais-Smale (PS) condition, if any sequence $\{u_n\} \in H$ with $J(u_n)$ bounded and $J'(u_n) \to 0$ has a convergent subsequence.*

2 Bound Estimates for Morse Index

Morse index provides understanding of the local structure of a critical point and is used as an instability index for an unstable solution. It is an important notion in stability analysis [4]. Although we have printed Morse index for each numerical solution computed by our minimax method in [10], their mathematical justifications have not been verified. In this section, we establish several bound estimates for the Morse index of a critical point based on our minimax method.

Lemma 2.1 *Let $v_0 \in S_{L^\perp}$ be a point. If there exist a neighborhood $\mathcal{N}(v_0)$ of v_0 and a locally defined mapping $p : \mathcal{N}(v_0) \cap S_{L^\perp} \to H$ such that $p(v) \in \{L, v\}$ $\forall v \in \mathcal{N}(v_0) \cap S_{L^\perp}$. If p is differentiable at v_0 and $u_0 = p(v_0) \notin L$, then*

$$p'(v_0)(\{L, v_0\}^\perp) + \{L, v_0\} = H.$$

Proof. For any $w \in \{L, v_0\}^\perp$, $\|w\| = 1$, denote $v_s = \frac{v_0+sw}{\|v_0+sw\|}$. Then there exists $s_0 > 0$ such that when $|s| < s_0$, we have $v_s \in \mathcal{N}(v_0) \cap S_{L^\perp}$.

Consider the one dimensional vector function $\alpha(s) = P_{L^\perp}(p(v_s))$, where $P_{L^\perp}$ is the projection onto $L^\perp$. Since p is differentiable at v_0 and v_s smoothly depends on s, α is differentiable at 0 and

$$\alpha'(0) = P_{L^\perp}(p'(v_0)(\frac{\partial v_s}{\partial s})) = P_{L^\perp}(p'(v_0)(w)).$$

On the other hand, $p(v_s) \in \{L, v_s\}$, we have $\alpha(s) = t_s v_s$, where $t_0 = \langle \alpha(s), v_s \rangle$ is differentiable. So $\alpha'(0) = t_0' v_0 + t_0 w$, where due to our assumption that $u_0 = p(v_0) \notin L$, we have $t_0 \neq 0$. The two different expressions of $\alpha'(0)$ imply

$$P_{L^\perp}(p'(v_0)(w)) = t_s'(0) v_0 + t_0 w.$$

Then it leads to $w \in \{p'(v_0)(w), L, v_0\}$. Since w is an arbitrary vector in $\{L, v_0\}^\perp$, it follows that

$$p'(v_0)(\{L, v_0\}^\perp) + \{L, v_0\} = H. \tag{2.1}$$

▮

Lemma 2.2 *Let $v_0 \in S_{L^\perp}$ be a point. If there exist a neighborhood $\mathcal{N}(v_0)$ of v_0 and a locally defined mapping $p : \mathcal{N}(v_0) \cap S_{L^\perp} \to H$ such that $p(v) \in \{L, v\}$ $\forall v \in \mathcal{N}(v_0) \cap S_{L^\perp}$. Assume that p is differentiable at v_0 and $u_0 = p(v_0) \notin L$. If u_0 is a critical point of J with $MI(u_0) > \dim L + 1$, then*

$$p'(v_0)(\{L, v_0\}^\perp) \cap H^- \neq \{0\}.$$

Proof. Denote H^- the negative subspace of $J''(u_0)$ and $k = \dim L + 1$. Then $\dim H^- > k$. By applying Lemma 2.1, there exit linearly independent vectors $e_0, e_1, \ldots, e_k \in H^-$ which can be represented as $e_i = g_i + f_i$ with $g_i \in p'(v_0)(\{L, v_0\}^\perp)$ and $f_i \in \{L, v_0\}$. $f_0, f_1, \ldots, f_k$ have to be linearly dependent because $k = \dim L + 1$. So we can find real numbers $a_0, a_1, \ldots, a_k$ such that $\sum_{i=0}^k a_i^2 \neq 0$ and $\sum_{i=0}^k a_i f_i = 0$. Therefore

$$\sum_{i=0}^k a_i e_i = \sum_{i=0}^k a_i g_i \in p'(v_0)(\{L, v_0\}^\perp) \cap H^-.$$

Because, $e_0, e_1, \ldots, e_k$ are linearly independent, $\sum_{i=0}^k a_i e_i \neq 0$. Thus, the conclusion of the lemma is verified. ▮

Theorem 2.1 *Let $v_0 \in S_{L^\perp}$ be a point. If J has a local peak selection p w.r.t. L at v_0 such that p is differentiable at v_0 and $u_0 = p(v_0) \notin L$. If v_0 is a local minimum point of $J \circ p$ on $S_{L^\perp}$, then u_0 is a critical point of J with $MI(u_0) \leq \dim L + 1$.*

Proof. Since p is a local peak selection of J w.r.t. L at v_0, there exists a neighborhood $\mathcal{N}(v_0)$ of v_0 such that $p(v) \in \{L, v\}$, $\forall v \in \mathcal{N}(v_0) \cap S_{L^\perp}$. By applying Lemma 2.1, we have

$$p'(v_0)(\{L, v_0\}^\perp) + \{L, v_0\} = H$$

or

$$codim(p'(v_0)(\{L, v_0\}^\perp)) \leq \dim L + 1.$$

Now suppose that $MI(u_0) > \dim L + 1$. Denote H^- the negative subspace of $J''(u_0)$. By Lemma 2.2, we have

$$p'(v_0)(\{L, v_0\}^\perp) \cap H^- \neq \{0\}. \tag{2.2}$$

Choose any $w \in \{L, v_0\}^\perp$, $\|w\| = 1$, such that $p'(v_0)(w) \in H^-$. Around $u_0 = p(v_0)$, we have the second Taylor expansion

$$J(u) = J(u_0) + \frac{1}{2}\langle J''(u_0)(u - u_0), u - u_0\rangle + o(\|u - u_0\|^2) \tag{2.3}$$

Denote $v_s = \frac{v_0 + sw}{\|v_0 + sw\|}$, we have $v_s \in \mathcal{N}(v_0) \cap S_{L^\perp}$ for $|s|$ small and then $\frac{dv_s}{ds} = w$. So it follows

$$p(v_s) = u_0 + sp'(v_0)(w) + o(|s|). \tag{2.4}$$

Combining the above two estimates (2.3) and (2.4), we obtain

$$\begin{aligned} & J(p(v_s)) \\ = \; & J(u_0) + \frac{1}{2}\langle J''(u_0)(sp'(v_0)(w) + o(|s|)), sp'(v_0)(w) + o(|s|)\rangle + o(\|sp'(v_0)(w) + o(|s|)\|^2) \\ = \; & J(u_0) + \frac{1}{2}s^2\langle J''(u_0)(p'(v_0)(w)), p'(v_0)(w)\rangle + o(s^2) \\ < \; & J(u_0), \end{aligned}$$

where the last strict inequality holds for $|s|$ sufficiently small, because $p'(v_0)(w) \in H^-$.

Since $v_s \in \mathcal{N}(v_0) \cap S_{L^\perp}$ and $u_0 = p(v_0)$, the above contradicts the assumption that v_0 is a local minimum point of $J \circ p$ on $S_{L^\perp}$. Therefore $MI(u_0) \leq \dim L + 1$. ∎

Theorem 2.2 *If p is a local peak selection of J w.r.t. L at $v_0 \in S_{L^\perp}$ and $u_0 = p(v_0)$ is a non-degenerate critical point of J, then $MI(u_0) \geq \dim L + 1$.*

Proof. Assume that $k = MI(u_0) < \dim L + 1$. By our assumption, u_0 is non-degenerate, i.e., $J''(u_0)$ is invertible, we have $H = H^+ \bigoplus H^-$ where H^+ is the maximum positive subspace and H^- is the maximum negative subspace corresponding to the spectral decomposition of $J''(u_0)$. It follows that $codim(H^+) = \dim(H^-) = k < \dim L + 1$, so there exists a non-zero vector $v \in H^+ \cap \{L, v_0\}$. When $v \in H^+$, for sufficient small t, we have $J(u_0 + tv) > J(u_0)$. But this contradicts to that u_0 is a local maximum point of J in the subspace $\{L, v_0\}$. Therefore, $MI(u_0) \geq \dim L + 1$. ▮

Theorem 2.3 *Assume that p is a local peak selection of J w.r.t. L at $v_0 \in S_{L^\perp}$ such that p is differentiable at v_0 and $u_0 = p(v_0) \notin L$. If v_0 is a local minimum point of $J \circ p$ on $S_{L^\perp}$, and $J''(u_0)$ is invertible, then $MI(u_0) = \dim L + 1$.*

Proof. Since under the conditions, we have proved that $u_0 = p(v_0)$ is a non-degenerate critical point of J. The conclusion follows by combining the last two theorems. ▮

Theorem 2.4 *Let $v_0 \in S_{L^\perp}$ be a point. If there exist a neighborhood $\mathcal{N}(v_0)$ of v_0 and a locally defined mapping $p: \mathcal{N}(v_0) \cap S_{L^\perp} \to H$ such that $p(v) \in \{L, v\}$, $J'(p(v)) \perp \{L, v\}$, $\forall v \in \mathcal{N}(v_0) \cap S_{L^\perp}$ and p differentiable at v_0. If $v_0 \in S_{L^\perp}$ is a local minimum point of $J \circ p$ on $S_{L^\perp}$ with $u_0 = p(v_0) \notin L$, then u_0 is a critical point of J with $MI(u_0) \leq \dim L + 1$.*

Proof. We first prove that $u_0 = p(v_0)$ is a critical point of J. The second part of the theorem follows from a similar proof of Theorem 2.1.

For any $w \in \{L, v_0\}^\perp$, denote

$$v(s) = \frac{v_0 + sw}{\|v_0 + sw\|}.$$

We have $v(s) \in \mathcal{N}(v_0) \cap S_{L^\perp}$ for $|s|$ small and $\frac{dv(s)}{ds} = w$. Therefore

$$\begin{aligned} p(v(s)) &= p(v_0) + sp'(v_0)\frac{dv(s)}{ds} + o(|s|) \\ &= u_0 + sp'(v_0)(w) + o(|s|). \end{aligned}$$

It follows that

$$\begin{aligned} J(p(v(s))) &= J(p(v_0)) + J'(p(v_0))(p(v(s)) - p(v_0)) + o(\|p(v(s)) - p(v_0)\|) \\ &= J(u_0) + sJ'(u_0)p'(v_0)(w) + o(|s|). \end{aligned}$$

If

$$J'(u_0)p'(v_0)(w) \neq 0$$

for some $w \in \{L, v_0\}^\perp$, then when $|s|$ is sufficiently small, we can choose either $s > 0$ or $s < 0$ such that

$$J(p(v(s))) < J(p(v_0))$$

which contradicts the assumption that v_0 is a local minimum point of $J \circ p$ on $S_{L^\perp}$. Thus

$$J'(u_0)p'(v_0)(\{L, v_0\}^\perp) = 0.$$

Since by our assumption

$$J'(u_0)(\{L, v_0\}) = 0$$

and by Lemma 2.1

$$p'(v_0)(\{L, v_0\}^\perp) + \{L, v_0\} = H,$$

it follows that

$$J'(u_0)u = 0 \quad \forall u \in H,$$

i.e., $u_0 = p(v_0)$ is a critical point of J. ∎

It is worthwhile indicating that if p is a local peak selection of J at $v_0 \in S_{L^\perp}$, then $p(v) \in [L, v]$ and $J'(p(v)) \perp [L, v]$ for all $v \in \mathcal{N}(v_0) \cap S_{L^\perp}$. If $\{v_n\} \subset S_{L^\perp}$, $v_n \to v_0$ and $u_n = p(v_n) \to u_0$, we have $u_0 \in [L, v_0]$ and $J'(u_0) \perp [L, v_0]$. So such a locally defined mapping generalized the notion of a local peak selection and resolved the problem that a limit of a sequence of local maximum points may not be a local maximum point. This generalization has a potential to design a new type of local algorithm for finding multiple saddle points that are not necessarily of a minimax type.

Theorem 2.5 *If $u_0 \notin L$ is a non-degenerate critical point of J such that u_0 is not a local minimum point of J along any direction $v \in \{L, u_0\}$, then*

$$MI(u_0) \geq \dim L + 1.$$

Proof. Assume that $k = MI(u_0) < \dim L + 1$. By our assumption, u_0 is non-degenerate, i.e., $J''(u_0)$ is invertible, we have $H = H^+ + H^-$ where H^+ is the maximum positive subspace and H^- is the maximum negative subspace corresponding to the spectral decomposition of $J''(u_0)$. It follows that $codim(H^+) = \dim(H^-) = k < \dim L + 1$, so there exists a non-zero vector $v \in H^+ \cap \{L, u_0\}$. When $v \in H^+$, for sufficiently small t, we have

$$J(u_0 + tv) > J(u_0).$$

It then contradicts the assumption that u_0 is not a local minimum point of J along any direction $v \in \{L, u_0\}$. ∎

3 Application to Semilinear Elliptic PDEs

Consider a semilinear elliptic Dirichlet BVP on a smooth bounded domain Ω in $\mathbb{R}^n$ which has many applications in physics, engineering, biology, ecology, geometry, etc

$$(3.1) \qquad \begin{cases} \Delta u(x) + f(x, u(x)) = 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

where the function $f(x, u(x))$ satisfies the following standard hypothesis:

(h1) $f(x, u(x))$ is locally Lipschitz on $\bar{\Omega} \times \mathbb{R}$;

(h2) there are positive constants a_1 and a_2 such that

$$(3.2) \qquad |f(x, \xi)| \leq a_1 + a_2 |\xi|^s$$

where $0 \leq s < \frac{n+2}{n-2}$ for $n > 2$. If $n = 2$,

$$(3.3) \qquad |f(x, \xi)| \leq a_1 \exp \phi(\xi)$$

where $\phi(\xi)\xi^{-2} \to 0$ as $|\xi| \to \infty$;

(h3) $f(x, u(\xi)) = o(|\xi|)$ as $\xi \to 0$;

(h4) there are constants $\mu > 2$ and $r \geq 0$ such that for $|\xi| \geq r$,

$$(3.4) \qquad 0 < \mu F(x, \xi) \leq \xi f(x, \xi),$$

where $F(x, \xi) = \int_0^\xi f(x, t)dt$.

(h4) says that f is superlinear, which implies that there exist positive numbers a_3 and a_4 such that for all $x \in \bar{\Omega}$ and $\xi \in \mathbb{R}$

$$F(x,\xi) \geq a_3|\xi|^\mu - a_4. \tag{3.5}$$

The variational functional associated to the Dirichlet problem (3.1) is

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx - \int_\Omega F(x,u(x))dx, \quad u \in H \equiv H_0^1(\Omega), \tag{3.6}$$

where we use an equivalent norm $\|u\| = \int_\Omega |\nabla u(x)|^2 dx$ for the Sobolev space $H = H_0^1(\Omega)$. It is well known [18] that under Conditions (h1) through (h4), J is C^1 and satisfy (PS) condition. A critical point of J is a weak solution, and also a classical solution of (3.1). 0 is a local minimum point of J and so a trivial solution. Moreover, in any finitely dimensional subspace of H, J goes to negative infinity uniformly. Therefore, for any finite dimensional subspace L, the peak mapping P of J w.r.t. L is nonempty.

We need one more hypothesis, that is

(h5) $\frac{f(x,\xi)}{|\xi|}$ is increasing w.r.t. ξ, or

(h5') $f(x,\xi)$ is C^1 w.r.t. ξ and $f_\xi(x,\xi) - \frac{f(x,\xi)}{\xi} > 0$.

It is clear that (h5') implies (h5). If $f(x,\xi)$ is C^1 in ξ, then (h5) and (h5') are equivalent. All the power functions of the form $f(x,\xi) = |\xi|^k\xi$ with $k > 0$, satisfies (h1) through (h5'), and so do all the positive linear combinations of such functions. Under (h5) or (h5'), J has only one local maximum point in any direction, or, the peak mapping P of J w.r.t. $L = \{0\}$ has only one selection. In other words, $P = p$. The proof can be found in [16] and [14].

Theorem 3.1 *Under the hypothesis (h1) through (h5), if the peak mapping P of J w.r.t. a finitely dimensional subspace L is singleton at $v_0 \in S_{L^\perp}$ and for any $v \in S_{L^\perp}$ around v_0, a peak selection $p(v)$ is a global maximum point of J in $[L,v]$, then p is continuous at v_0.*

Proof. See [10]. ∎

Theorem 3.2 *Assume that Conditions (h1) – (h5') are satisfied and that there exist positive constants a_5 and a_6 s.t.*

$$|f_\xi(x,\xi)| \leq a_5 + a_6|\xi|^{s-1} \tag{3.7}$$

where s is specified in (h2). Then the only peak selection p of J w.r.t. $L = \{0\}$ is C^1.

Proof. See [10]. ∎

Since $w_0 = 0$ is the local minimum point of J and along each direction $v \in H$, J has only one maximum point $p(v)$, we have $J(p(v)) > 0, \forall v \in S$. If for each $v \in S_{L^\perp}$, $p(v)$ is a local maximum point of J in $[L, v]$, then $p(v)$ is the only local maximum point of J along the direction $u = \frac{p(v)}{\|p(v)\|}$, Therefore we have $J(p(v)) > 0, \forall v \in S_{L^\perp}$. As a composite function $J(p(\cdot))$ is bounded from below by 0. So Ekeland's variational principle can be applied. With the PS condition, existence result can also be established.

Theorem 3.3 *Under the hypothesis of (h1) through (h5), and that there exist positive constants a_5 and a_6 such that*

$$|f_\xi(x, \xi)| \leq a_5 + a_6|\xi|^{s-1},$$

where s is specified in (h2), if $v_0 = \arg\min_{v \in S_H} J(p(v))$ then $u_0 = p(v_0)$ is a critical point with $MI(u_0) = 1$.

Proof. Assume $u_0 = p(v_0) = t_0 v_0$. Consider the 1-dimensional function

$$g(t) = J(tv_0) = \frac{1}{2}\int_\Omega t^2|\nabla v_0(x)|^2\, dx - \int_\Omega F(x, tv_0(x))\, dx.$$

We have

$$g'(t) = t\int_\Omega |\nabla v_0(x)|^2\, dx - \int_\Omega f(x, tv_0(x))v_0(x)\, dx.$$

So

$$1 = \int_\Omega \frac{f(x, t_0 v_0(x))}{t_0 v_0(x)} v_0^2(x)\, dx.$$

Meanwhile we have

$$g''(t) = \int_\Omega |\nabla v_0(x)|^2\, dx - \int_\Omega f_\xi(x, tv_0(x))v_0^2(x)\, dx$$

$$\begin{aligned} g''(t_0) &= 1 - \int_\Omega f_\xi(x, t_0 v_0(x))v_0^2(x)\, dx \\ &< 1 - \int_\Omega \frac{f(x, t_0 v_0(x))}{t_0 v_0(x)} v_0^2(x)\, dx \quad \text{(ref. (h5)} \\ &= 0, \end{aligned}$$

which implies that

$$H^0 \cap \{L, v_0\} = \{0\},$$

where H^0 is the nullspace of J'' at u_0 and $L = \{0\}$. By Theorem 2.1, we obtain

$$\text{MI}(u_0) = \dim(L) + 1 = 1.$$

4 Some New Saddle Point Theorems

As our convergence results in [11] indicate that our algorithm can be used to find a non-minimax critical point, e.g., a Monkey saddle point. Thus the argument already exceeded the scope of a minimax approach. So far the only results we found in the critical point theory which are more general than a minimax principle are two theorems proved by S. I. Pohozaev in [9] or [17]. The following results are interesting generalizations. The first one is an embedding result. It is general but lacks of characterization. The second result has potential applications in devising a new numerical algorithm.

Theorem 4.1 *Given $L = span\{w_1, ..., w_k\}$ in H and let*

$$\bar{J}(t, v, t_1, .., t_k) \equiv J(tv + t_1 w_1 + ... + t_k w_k).$$

If $(t_0^*, v^*, t_1^*, .., t_k^*)$ is a conditional critical point of $\bar{J}$ subject to $v \in S_{L^\perp}$ with $t^* \neq 0$, then $t_0^* v^* + t_1^* w_1 + ... + t_k^* w_k$ is a critical point of J.

Proof: By the Lagrange Multiplier Theorem, there exist $\lambda, \mu, \eta_1, .., \eta_k$ with $\lambda^2 + \mu^2 + \eta_1^2 + ... + \eta_k^2 \neq 0$ such that the Lagrange functional

$$\mathcal{L}(t, v, t_1, ..., t_k) = \lambda \bar{J}(t, v, t_1, ..., t_k) + \mu \|v\| + \sum_{i=1}^{k} \eta_i \langle w_i, v \rangle$$

has a critical point at $(t_0^*, v^*, t_1^*, .., t_k^*)$. So we have

(4.8) $$\frac{\partial \mathcal{L}}{\partial t} = 0 \quad \Rightarrow \quad \lambda \bar{J}_t(t_0^*, v^*, t_1^*, ..., t_k^*) = 0;$$

(4.9) $$\frac{\partial \mathcal{L}}{\partial v} = 0 \quad \Rightarrow \quad \lambda \bar{J}'_v(t_0^*, v^*, t_1^*, ..., t_k^*) + \mu \|v^*\|' + \sum_{i=1}^{k} \eta_i w_i = 0;$$

$$\frac{\partial \mathcal{L}}{\partial t_i} = 0 \quad \Rightarrow \quad \lambda \bar{J}'_{t_i}(t_0^*, v^*, t_1^*, .., t_k^*) = 0 \quad \text{or}$$

(4.10) $$\lambda \langle J'(t_0^* v^* + t_1^* w_1 + ... + t_k^* w_k), w_i \rangle = 0, \ (i = 1, .., k).$$

From (4.9), we have

$$\langle \lambda \bar{J}_v(t_0^*, v^*, t_1^*, .., t_k^*), v\rangle + \mu\langle \|v^*\|', v\rangle + \sum_{i=1}^{k} \eta_i \langle w_i, v\rangle = 0 \quad \forall v \in H. \tag{4.11}$$

In particular

$$\lambda\langle \bar{J}_v(t_0^*, v^*, t_1^*, .., t_k^*), v^*\rangle + \mu\langle \|v^*\|', v^*\rangle + \sum_{i=1}^{k} \eta_i \langle w_i, v^*\rangle = 0. \tag{4.12}$$

Since $\langle \|v^*\|', v^*\rangle = 1$ and $\langle w_i, v^*\rangle = 0$ for $i = 1, ..., k$, we obtain

$$\lambda\langle \bar{J}_v(t_0^*, v^*, t_1^*, .., t_k^*), v^*\rangle + \mu = 0. \tag{4.13}$$

So $\lambda = 0$ will lead to $\mu = 0$ and then $\eta_i = 0$ by choosing $v = w_i$ in (4.10). It contradicts $\lambda^2 + \mu^2 + \eta_1^2 + ... + \eta_k^2 \neq 0$. Therefore $\lambda \neq 0$ and (4.8) gives

$$\bar{J}_t'(t_0^*, v^*, t_1^*, ..., t_k^*) = 0$$

or

$$\langle J'(t_0^* v^* + t_1^* w_1 + ... + t_k^* w_k), v^*\rangle = 0.$$

It leads to

$$\langle \bar{J}_v'(t_0^*, v^*, t_1^*, ..., t_k^*), v^*\rangle = t_0^*\langle J'(t_0^* v^* + t_1^* w_1 + ... + t_k^* w_k), v^*\rangle = 0.$$

(4.13) then yields $\mu = 0$. Taking any $v \perp [w_1, ..., w_k]$ in (4.11), we obtain

$$\lambda\langle \bar{J}_v'(t_0^*, v^*, t_1^*, ..., t_k^*), v\rangle = 0$$

or

$$t_0^*\langle J'(t_0^* v^* + t_1^* w_1 + ... + t_k^* w_k), v\rangle = 0.$$

Since $t_0^* \neq 0$, it leads to

$$\langle J'(t_0^* v^* + t_1^* w_1 + ... + t_k^* w_k), v\rangle = 0 \quad \forall v \perp [w_1, ..., w_k].$$

Taking (4.10) into account, we have

$$J'(t_0^* v^* + t_1^* w_1 + ... + t_k^* w_k) = 0. \qquad \blacksquare$$

Theorem 4.1 reduces to Pohozaev's embedding result in [9] or [17] if we set $L = \{0\}$.

Theorem 4.2 *Let $v^* \in S_{L^\perp}$ be a point. If J has a local peak selection p w.r.t. L at v^* and $u^* = p(v^*)$ such that*

(i) p is Lipschitz continuous at v^*,

(ii) $dis(u^*, L) > 0$,

(iv) u^* is a conditional critical point of J on $p(S_{L^\perp})$,

then u^* is a critical point of J.

Proof: Suppose that $\|J'(u^*)\| > 0$. Set $\delta = \frac{1}{2}\|J'(u^*)\|$. By Lemma 1.1, there exists $s_0 > 0$ such that

$$J(p(v^*(s))) - J(p(v^*)) < -\delta dis(u^*, L)\|v^*(s) - v^*\| \quad \forall 0 < s < s_0$$

where

$$v^*(s) = \frac{v^* + sd}{\|v^* + sd\|} \in \mathcal{N}(v^*) \cap S_{L^\perp}, \quad d = -J'(p(v^*)).$$

Then we have

$$\text{(4.14)} \qquad \frac{J(p(v^*(s))) - J(p(v^*))}{\|p(v^*(s)) - u^*\|} \frac{\|p(v^*(s)) - u^*\|}{\|v^*(s) - v^*\|} < -\delta dis(u^*, L), \quad \forall\, 0 < s < s_0,$$

where $\mathcal{N}(v^*)$ is a neighborhood of v^* in which the local peak selelction p is defined. Since p is Lipschitz continuous at v^* and u^* is a conditional critical point of J on $p(S_{L^\perp})$,

$$\frac{\|p(v^*(s)) - u^*\|}{\|v^*(s) - v^*\|}$$

is bounded and

$$\frac{J(p(v^*(s))) - J(p(v^*))}{\|p(v^*(s)) - u^*\|} \to 0 \text{ as } s \to 0.$$

So the left hand side of (4.14) goes to zero, which leads to a contradiction. ∎

Theorem 4.3 *Let $v_0 \in S_{L^\perp}$ be a point. If J has a local peak selection p w.r.t. L such that p is differentiable at v_0 and $u_0 = p(v_0) \notin L$. If u_0 is a conditional critical point of J on $p(S_{L^\perp})$ and v_0 is not a local maximum point of $J \circ p$ along the projection of any direction v on $S_{L^\perp}$, then u_0 is a critical point of J with $MI(u_0) \leq \dim L + 1$. In addition, if u_0 is nondegenerate, then $MI(u_0) = \dim L + 1$.*

Proof. By Theorem 4.2, we obtain that u_0 is a critical point of J. Then following a similar argument in the proof of Theorem 2.1, until we have

$$\begin{aligned}
& J(p(v_s)) \\
= \;& J(u_0) + \frac{1}{2}\langle J''(u_0)(sp'(v_0)(w) + o(|s|)), sp'(v_0)(w) + o(|s|)\rangle + o(\|sp'(v_0)(w) + o(|s|)\|^2) \\
= \;& J(u_0) + \frac{1}{2}s^2\langle J''(u_0)(p'(v_0)(w)), p'(v_0)(w)\rangle + o(s^2) \\
< \;& J(u_0),
\end{aligned}$$

where

$$v_s = \frac{v_0 + sw}{\|v_0 + sw\|} \in \mathcal{N}(v_0) \cap S_{L^\perp} \quad \text{and} \quad w \in [L, v_0]^\perp, \|w\| = 1.$$

So the last strict inequality contradicts to our assumption that v_0 is not a local maximum point of $J \circ p$ along the projection of any direction v on $S_{L^\perp}$. Thus $MI(u_0) \le \dim L + 1$. If in addition, u_0 is nondegenerate, we can apply Theorem 2.5 to conclude that $MI(u_0) = \dim L + 1$. ∎

Condition (iv) in Theorem 4.2 is clearly satisfied if w^* is a local minimum point of J on the solution (stable) manifold $\mathcal{M} = p(S_{L^\perp})$. So it is clear that Theorems 4.1 and 4.2 are indeed more general than a minimax principle. As matter of fact, Condition (iv) in Theorems 4.2 can be weekend as that w^* is a conditional critical point of J on any subset containing the path $p(v^*(s))$ for small $s \ge 0$.

If we set $L = \{0\}$, the trivial subspace and assume $p(v)$ is the global maximum point of J on $[L, v] = \{tv : t > 0\}$ for each $v \in S_{L^\perp} = S_H$ and p is C^1, then Theorem 4.2 reduces to a result of Pohozaev in [17]. As we did in our algorithm, by gradually expanding the subspace L, Theorem 4.2 can be used to provide us with information on the Morse index of the critical point. For an example, when we solve a semilinear elliptic equation as shown in [10], we start from the trivial solution and set $L = \{0\}$ to approximate a solution w_1 with MI= 1; then we set $L = \{w_1\}$ to search for a solution w_2 with MI= 2,.... This is the advantage of our approach.

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