A Local Min-Orthogonal Method for Finding Multiple Saddle Points

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Abstract

The purpose of this paper is twofold. The first is to remove a possible ill-posedness related to a local minimax method developed in [10,11] and the second is to provide a local characterization for non-minimax type saddle points. To do so, a local L-$\perp$ selection is defined and a necessary and sufficient condition for a saddle point is established, which leads to a min-orthogonal method. Those results exceed the scope of a minimax principle, the most popular approach in critical point theory. An example is given to illustrate the new theory. With this local characterization, the local minimax method in [10,11] is generalized to a local min-orthogonal method for finding multiple saddle points. In a subsequent paper [23], this approach is applied to define a modified pseudo gradient (flow) of a functional for finding multiple saddle points in Banach spaces.

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Abbreviated titles. A Local Min-Orthogonal Method

1 Introduction

Let H be a Hilbert space, $\langle \cdot, \cdot \rangle$ be its inner product and $J : H \to \mathbb{R}$ be a Frechet differentiable functional, called a *generic energy function*. Denote by δJ its Frechet derivative, J' its gradient, i.e., for each $u \in H$, $J'(u)$ is the unique point in H such that $\langle J'(u), v\rangle = \delta J(u)v = \frac{d}{dt}J(u+tv)|_{t=0}$, $\forall v \in H$, and J'' its second Frechet derivative if it exists. A point $u^* \in H$ is called a *critical point* of J if u^* solves the Euler-Lagrange equation $J'(u^*) = 0$ or $\langle J'(u), v\rangle = 0$, $\forall v \in H$. A critical point u^* is said to be *non-degenerate* if $J''(u^*)$ exists and is invertible. Otherwise u^* is said to be *degenerate*. The first candidates for a critical point are the local maxima and minima to which the classical critical point theory was devoted in calculus of variation. Traditional numerical methods focus on finding such stable solutions. Critical points that are not local extrema are *unstable* and called *saddle points*. In physical systems, saddle points appear as unstable equilibria or transient excited states.

The *Morse Index* of a critical point u^* is $\mathrm{MI}(u^*) = \dim(H^-)$ where H^- is the maximum negative definite subspaces of $J''(u^*)$ in H. Thus for a non-degenerate critical point, if MI $= 0$, then it is a local minimizer and a stable solution, and if $MI > 0$, then it is a min-max type and unstable solution.

Multiple solutions with different performance, maneuverability and instability properties exist in many nonlinear problems in applications [20, 18, 13, 22, 12]. Vector solitons arise in many fields, such as condensed matter physics, dynamics of biomolecules, nonlinear optics, etc. For example, in the study of self-guided light waves in nonlinear optics [8,9,14], excited states are of great interests. All those solitons are saddle points, thus unstable solutions. Among them, solutions which are not ground states, are the so-called excited states. Among many different modes of excited states are the vortex-mode and dipole-mode vector solitons, it has been experimentally and numerically proved that those two unstable solutions have very different instability and maneuverability properties. The vortex-mode can be easily perturbed to decay into a dipole-mode. While the dipole-modes are much more stable, "stable enough for experimental observation, ..., extremely robust, have a typical lifetime of several hundred diffraction lengths and survive a wide range of perturbations"[8], thus hard to excite.

It is interesting for both theory and applications to numerically solve for multiple solutions in a stable way. Before [10], little is known in the literature to devise such a feasible numerical algorithm. Minimax principle is one of the most popular approaches in critical point theory. However, most minimax theorems in the literature (See [1, 7, 16, 17, 18, 19, 22]) are in the *global theory*, which focus basically on the existence issue, such as the well-known mountain pass lemma, various linking and saddle point theorems. They require one to solve a two-level *global* optimization problem and therefore not for algorithm implementation. On the other hand, the *local theory* which studies the local characterization, local behavior and local instability of critical points has not been developed.

In [10], motivated by the numerical works of Choi-McKenna [4] and Ding-Costa-Chen [6] and the idea to define a solution manifold [15, 5], a new local minimax method which characterizes a saddle point as a solution to a two-level *local* minimax problem, is developed. The basic idea of the method is to define a local peak selection [10, 24].

Let H be a Hilbert space and $L \subset H$ be a closed subspace, called a *support*. Denote $S_{L^\perp} = \{v \in L^\perp : \|v\| = 1\}$ and $\{L, v\} = \{tv + v_L : t \in \mathbb{R}, v_L \in L\}$ for each $v \in S_{L^\perp}$. A set-valued mapping $P\colon S_{L^\perp} \to 2^H$ is called a *peak mapping* of J if $P(v)$ is the set of all local maximum points of J in $\{L, v\}$. A single-valued mapping $p\colon S_{L^\perp} \to H$ is called a *peak selection* if $p(v) \in P(v)$ $\forall v \in S_{L^\perp}$. If there are a point $v \in S_{L^\perp}$ and a neighborhood $\mathcal{N}(v)$ of v, such that P (p) is locally defined in $\mathcal{N}(v) \cap S_{L^\perp}$, then P (p) is called a *local peak mapping (selection)* of J w.r.t. L at v.

The following theorem characterizes a saddle point as a local minimax solution which laid a mathematical foundation for the local minimax method [10] for finding multiple critical points. To the best of our knowledge, it is the first local minimax theorem established in critical point theory.

Theorem 1.1. *[10] Let $v_0 \in S_{L^\perp}$ and p be a local peak selection J w.r.t. L at v_0 such that (a) p is continuous at v_0, (b) $dis(p(v_0), L) > 0$ and (c) v_0 is a local minimum point of $J(p(v))$ on $S_{L^\perp}$. Then $p(v_0)$ is a critical point of J.* ∎

If we define a solution set

$$\mathcal{M} = \{p(v) : v \in S_{L^\perp}\}, \tag{1.1}$$

in a neighborhood of v_0, then the above theorem states that a local minimum point of J on $\mathcal{M}$ yields a saddle point. A local minimum point of $J(p(v))$ can be numerically approximated by, e.g., a steepest descent method, which leads to the numerical local minimax algorithm [10] for finding multiple critical points is devised. The numerical method has been successfully implemented to solve many semilinear elliptic PDE on various domains for multiple solutions [10, 11]. Some convergence results of the method are established in [11]. In [24], the local minimax method is used to define an index to measure the instability of a saddle point which can be computationally carried out. To be more specific, we have

THEOREM 1.2. *(Theorem 2.5 in [24]) Let $v^* \in S_{L^\perp}$. Let p be a local peak selection of J w.r.t. L at v^* s.t. p is differentiable at v^*, $u^* \equiv p(v^*) \notin L$ and $v^* = \arg\min_{v \in S_{L^\perp}} J(p(v))$. Then u^* is a critical point with* $\dim(L) + 1 = MI\,(u^*) + \dim(H^0 \cap \{L, v^*\})$. ▮

The following questions motivate us for this work:

(1) In the above results for the local minimax method, it is assumed that a local peak selection p is continuous or differentiable at v^*. How to check this condition? It is very difficult, since p is not explicitly defined. In particular, the graph of p or the solution set $\mathcal{M}$ is, in general, not closed, i.e., a limit of a sequence of local maximum points is not necessarily a local maximum point. In other words, if $v_n \to v^*$, $p(v_n)$ is a local maximum point of J in $\{L, v_n\}$ and $p(v_n) \to \bar{u}$, but $\bar{u}$ may not be a local maximum point of J in $\{L, v^*\}$. Thus p may not be defined at v^* as a peak selection. How can we say that p is continuous at v^*? This is also an ill condition as long as numerical computation is concerned;

(2) Non-minimax type saddle points, such as the monkey saddles do exist [3], see Fig. 1. It is known that all non-minimax type saddle points are degenerate. Due to degeneracy, Morse theory cannot handle them and minimax principle, one of the most popular approaches in critical point theory can not cover them either. How to establish a mathematical approach to cover those non-minimax type saddle points?

By analysis, we find that those two questions are closely related to the notion of a peak selection or the solution set. To answer the questions, in this paper, we develop a new and more general method by generalizing the definition of a peak selection so that the corresponding solution set $\mathcal{M}$ is closed and contains the solution set defined by a peak selection as a subset.

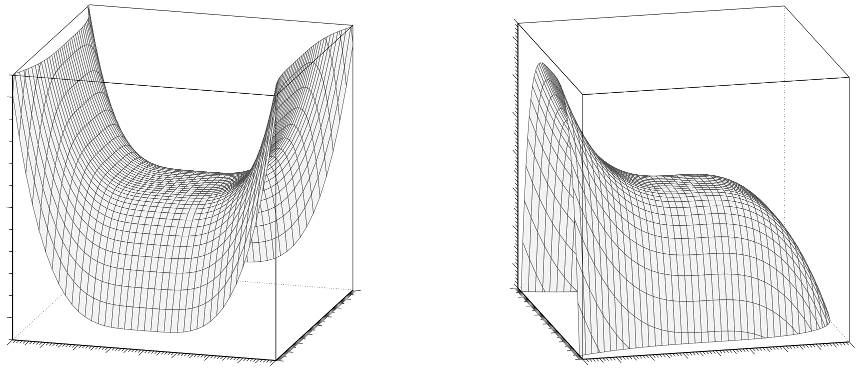

FIG. 1. *A horse saddle (minimax type) (left) and a monkey saddle (non-minimax type) (right).*

2 A New Local Characterization of Saddle Points

To study a dynamic problem, Nehari [15] introduced the concept of a solution manifold $\mathcal{M}$, and proved that a global minimizer of the energy functional on $\mathcal{M}$ yields a solution to the underlying dynamic problem (with MI = 1). Ding-Ni [5] generalized Nehari's idea in studying the following semilinear elliptic boundary value problem

$$\Delta u(x) + f(u(x)) = 0 \quad x \in \Omega, \quad u \in H = H_0^1(\Omega) \tag{2.1}$$

where Ω is a bounded open domain in $\mathbb{R}^N$, $f(t)$ is a nonlinear function satisfying $f'(t) > \frac{f(t)}{t}, t \neq 0$ and other standard conditions and $\langle u, v\rangle_{H^1(\Omega)} = \int_\Omega \nabla u \cdot \nabla v \, dx, \ \forall u, v \in H$. The associated variational functional is the energy function

$$J(u) = \int_\Omega \{\frac{1}{2}|\nabla u(x)|^2 - F(u(x))\}\, dx, \quad u \in H \quad \text{where} \quad F(t) = \int_0^t f(\tau) d\tau. \tag{2.2}$$

Then a direct computation shows that solutions to the BVP (2.1) coincide with critical points of J in H. Ding-Ni defined a solution manifold

$$\mathcal{M} = \left\{ v \in H_0^1(\Omega) | v \neq 0, \int_\Omega \left[|\nabla v|^2 - v f(v) \right] \, dx = 0 \right\}, \tag{2.3}$$

and proved [5] that a global minimizer of the energy function J on $\mathcal{M}$ yields a (ground state) solution (with MI = 1) to (2.1). Here the solution manifold (2.3) is used to describe a condition for a point $u = tv$ where $\|v\| = 1$ to be a maximum point (unique under the condition $f'(t) > \frac{f(t)}{t}, t \neq 0$) of J along the direction $v \in H$. It is clear that this is a special case of the solution set $\mathcal{M}$ with $L = \{0\}$ in (1.1). Now through integration-by-parts, Ding-Ni's solution manifold can be rewritten as

$$\mathcal{M} = \left\{ v \in H_0^1(\Omega) | v \neq 0, \int_\Omega [\Delta v + f(v)]\, v \, dx = 0 \right\}. \tag{2.4}$$

Recall for each $u \in H_0^1(\Omega)$, $J'(u) \in H_0^1(\Omega)$ is defined by, for each $v \in H_0^1(\Omega)$,

$$\langle J'(u), v\rangle_{H_0^1(\Omega)} = \frac{d}{d\alpha}\Big|_{\alpha=0} J(u+\alpha v) = \int_\Omega \{\nabla u \nabla v - f(u)v\}dx = -\int_\Omega (\Delta u + f(u))v\,dx.$$

Thus (2.4) can be expressed as

$$\mathcal{M} = \{v \in H_0^1(\Omega) | v \neq 0, \langle J'(v), v\rangle_{H_0^1(\Omega)} = 0\}.$$

It becomes an orthogonal condition. This observation and our idea to use a support L to define a peak mapping inspire us for the following generalized definitions.

DEFINITION 2.1. *Let L be a closed subspace of H. A set-valued mapping $P\colon S_{L^\perp} \to 2^H$ is called an L-$\perp$ mapping of J if*

$$P(v) = \{u \in \{L, v\} : J'(u) \perp \{L, v\}\} \quad \forall v \in S_{L^\perp}.$$

A mapping $p\colon S_{L^\perp} \to H$ is called an L-$\perp$ selection of J if $p(v) \in P(v)$ $\forall v \in S_{L^\perp}$. Let $v \in S_{L^\perp}$ and $\mathcal{N}(v)$ be a neighborhood of v. If P (p) is locally defined in $\mathcal{N}(v) \cap S_{L^\perp}$, then P (p) is called a local L-$\perp$ mapping (selection) of J at v.

It is clear that if u is a local maximum point of J in $\{L, v\}$, then u is an L-$\perp$ point of J in $\{L, v\}$ as well. Thus Definition 2.1 generalizes the notion of a peak mapping (selection). The solution set is now defined by $\mathcal{M} = \{p(v) : v \in S_{L^\perp}\}$.

Note that the graph of P can be very complicated, it may contain multiple branches, U-turn or bifurcation points. We will show that such defined L-$\perp$ selection has several interesting properties.

LEMMA 2.1. *If J is C^1, then the graph $G = \{(u, v) : v \in S_{L^\perp}, u \in P(v) \neq \emptyset\}$ is closed.*

Proof. Let $(u_n, v_n) \in G$ and $(u_n, v_n) \to (u_0, v_0)$. We have $u_n \in \{L, v_n\}$ and $J'(u_n) \perp \{L, v_n\}$. Since $u_n = t_n v_n + v_n^L \to u_0$ for some scalars t_n and points $v_n^L \in L$. Denote $u_0 = u_0^\perp + u_0^L$ for some $u_0^\perp \in L^\perp$ and $u_0^L \in L$. It follows $\|u_n - u_0\|^2 = \|t_n v_n - u_0^\perp\|^2 + \|v_n^L - u_0^L\|^2 \to 0$, i.e., $t_n v_n \to u_0^\perp = t_0 v_0$ for some scalar t_0 and $v_n^L \to u_0^L \in L$, because $v_n \to v_0$, $v_n^L \in L$ and L is closed. Thus $u_n \to u_0 = t_0 v_0 + u_0^L \in \{L, v_0\}$ and $J'(u_0) \perp \{L, v_0\}$ because J is C^1. Therefore $v_0 \in S_{L^\perp}$ and $u_0 \in P(v_0)$, i.e., $(u_0, v_0) \in G$. ∎

Now the ill-condition for a local peak selection has been removed.

DEFINITION 2.2. *Let $v^* \in S_{L^\perp}$ and p a local L-$\perp$ selection of J at v^*. For $u^* = p(v^*) \in L$, we say that u^* is an* isolated *L-$\perp$ point of J w.r.t. p if there exist neighborhoods $\mathcal{N}(u^*)$ of*

u^* and $\mathcal{N}(v^*)$ of v^* such that

$$\mathcal{N}(u^*) \cap L \cap p(\mathcal{N}(v^*) \cap S_{L^\perp}) = \{u^*\},$$

i.e., for each $v \in \mathcal{N}(v^*) \cap S_{L^\perp}$ and $v \neq v^*$ either $p(v) \notin L$ or $p(v) = u^*$.

THEOREM 2.1. *Let $v^* \in S_{L^\perp}$ and p be a local L-$\perp$ selection of J at v^* and continuous at v^*. Assume either $p(v^*) \notin L$ or $p(v^*) \in L$ is an isolated L-$\perp$ point of J w.r.t. p, then a necessary and sufficient condition that $u^* = p(v^*)$ is a critical point of J is that there exists a neighborhood $\mathcal{N}(v^*)$ of v^* such that*

$$J'(p(v^*)) \perp p(v) - p(v^*), \quad \forall v \in \mathcal{N}(v^*) \cap S_{L^\perp}. \tag{2.5}$$

Proof. Only need to prove the sufficiency. Since $J'(u^*) \perp \{L, v^*\}$, it suffices to show $J'(u^*) \perp L^\perp$. Let $\mathcal{N}(v^*)$ be a neighborhood of v^* such that p is defined and (2.5) is satisfied. If $p(v^*) \notin L$, i.e., $u^* \equiv p(v^*) = t^* v^* + v_L^*$ for some $t^* \neq 0$ and $v_L^* \in L$, then by the continuity, $p(v) = t_v v + v_L$ with $t_v \neq 0$ and $v_L \in L$ for each $v \in \mathcal{N}(v^*) \cap S_{L^\perp}$. Since $J'(u^*) \perp \{L, v^*\}$ and $p(v^*) \in \{L, v^*\}$, we have

$$J'(u^*) \perp p(v) - p(v^*) \Longleftrightarrow J'(u^*) \perp p(v) \Longleftrightarrow J'(u^*) \perp v, \quad \forall v \in \mathcal{N}(v^*) \cap S_{L^\perp}.$$

The above is equivalent to $J'(u^*) \perp v, \quad \forall v \in L^\perp$, since for each $v \in L^\perp$, when $|s|$ is small

$$\frac{v^* + sv}{\|v^* + sv\|} \in \mathcal{N}(v^*) \cap S_{L^\perp}.$$

Next if $u^* \equiv p(v^*) \in L$ is an isolated critical point of J relative to L and p, for each $v \in L^\perp$ and $|s| > 0$ small, consider

$$v^*(s) \equiv \frac{v^* + sv}{\|v^* + sv\|} \in \mathcal{N}(v^*) \cap S_{L^\perp} \quad \text{and} \quad p(v^*(s)) = t_s v^*(s) + v_L^*(s)$$

for some $v_L^*(s) \in L$. If $t_s \neq 0$, similar to the above, we have

$$J'(u^*) \perp p(v^*(s)) - p(v^*) \Longleftrightarrow J'(u^*) \perp v.$$

If $t_s = 0$, i.e., $u_s \equiv p(v^*(s)) = v_L^*(s) \in L$, since u^* is an isolated critical point of J relative to L and p, and p is continuous at v^*, we obtain $u_s = u^*$ when $|s| > 0$ is small. It follows that $J'(u^*) = J'(u_s) = J'(p(v^*(s)))$ is orthogonal to $\{L, v^*, v^*(s)\}$ and then

$$J'(u^*) \perp v^*(s) \Longleftrightarrow J'(u^*) \perp v, \quad \forall v \in L^\perp. \quad \blacksquare$$

Remark 2.1. *Theorem 2.1 is so far the most general characterization of multiple saddle points. It shows that the nature of local characterization of a critical point is not about minimum or maximum, it is about orthogonality. Also observe that except for J', J is not involved in the theorem or its proof. This implies that the result still holds true for non variational problem. Replacing J' by an operator $A : H \to H$ in the definition of a local L-$\perp$ selection p, Theorem 2.1 provides a necessary and sufficient condition that $u^* = p(v^*)$ solves $A(u) = 0$, a potentially useful result in solving multiple solutions to non variational problems.*

When a problem is variational, one way to satisfy the orthogonal condition in (2.5) is to look for a local minimum point v^* of $J(p(v))$. We are now dealing with a composite function $J(p(v))$. The main reason we use a composite function $J(p(v))$ rather than $J(v)$ is that we try to find multiple solutions. The operator p is used to stay away from old solutions. For example, when a peak selection p is used, $p(v)$ is a local maximum point of J in $\{L, v\}$ where L is spanned by previously found solutions. Thus it can usually be expected that $p(v) \notin L$. To find a local minimum of $J(p(v))$, we need to discuss a descent direction of a composite function. Let $u = \phi(v)$ be a locally defined smooth mapping. Write $\mathcal{J}(v) \equiv J(\phi(v))$. Then

$$\delta\mathcal{J}(v) = \delta J(u)\delta\phi(v).$$

Thus $u^* = \phi(v^*)$ is a critical point of J implies that v^* is a critical point of $\mathcal{J}$. But we are interested in the reversion, i.e., under what condition that v^* is a critical point of $\mathcal{J}$ will imply that $u^* = \phi(v^*)$ is a critical point of J? Then a critical point v^* of $\mathcal{J}$ can be found, for example, by a local minimization process.

The following lemma presents an interesting property enjoyed by a local L-$\perp$ selection. Since the proof can follow along the same line as in Lemma 2.3 in [24], it is omitted here.

Lemma 2.2. *(Lemma 2.3 in [24]) Let $v^* \in S_{L^\perp}$ and p be a local L-$\perp$ selection of J at v^*. If p is differentiable at v^* and $u^* = p(v^*) \notin L$, then*

$$\delta p(v^*)(\{L, v^*\}^\perp) \oplus \{L, v^*\} = H. \quad \blacksquare \tag{2.6}$$

Theorem 2.2. *Let p be a local L-$\perp$ selection of J at $v^* \in S_{L^\perp}$ and $\mathcal{J}(v) = J(p(v))$. If v^* is a critical point of $\mathcal{J}$, p is differentiable at v^* and $u^* = p(v^*) \notin L$, then $u^* = p(v^*)$ is a critical point of J.*

Proof. By the definition, we have $J'(u^*) = J'(p(v^*)) \perp \{L, v^*\}$ or $\delta J(p(v^*))(\{L, v^*\}) = 0$. Since $0 = \delta \mathcal{J}(v^*) = \delta J(u^*)\delta p(v^*)$, we have $\delta J(u^*)\delta p(v^*)(\{L, v^*\}^{\perp}) = 0$. Taking (2.6) into account, we have $\delta J(u^*)v = 0 \quad \forall v \in H$, i.e., u^* is a critical point of J. ∎

For example, when the function J on $H = H_0^1(\Omega)$ is given by (2.2), $d = -J'(u)$ is defined by

$$\begin{aligned} \langle J'(u), v\rangle_H &= \delta J(u)v = \frac{d}{d\alpha}\Big|_{\alpha=0} J(u+\alpha v) = -\int_\Omega (\Delta u(x) + f(u(x)))v(x)\,dx \\ &= -\langle d, v\rangle_H \equiv -\int_\Omega \nabla d(x)\cdot\nabla v(x)\,dx = \int_\Omega \Delta d(x) v(x)\,dx \quad \forall v \in H. \end{aligned}$$

Therefore $d = -J'(u)$ is solved from the linear elliptic PDE

$$\left\{ \begin{aligned} \Delta d(x) &= -\Delta u(x) - f(u(x)), x \in \Omega, \\ d(x) &= 0, x \in \partial\Omega. \end{aligned} \right.$$

For $d \in H$

$$v(s) = v + sd, \quad s > 0 \tag{2.7}$$

defines a linear variation at v in the direction d with stepsize s. d is said to be a descent direction of J at v if for $s > 0$ small, $J(v(s)) < J(v)$, or equivalently

$$\langle J'(v), v(s) - v\rangle = s\langle J'(v), d\rangle < 0 \quad \text{i.e.} \quad \langle J'(v), d\rangle < 0. \tag{2.8}$$

In particular, $d = -J'(v)$ is a descent direction. A unit vector $d \in H$ is said to be a steepest descent direction of J at $u \in H$ if

$$\langle J'(u), d\rangle_H = \min_{v\in H, \|v\|=1} \langle J'(u), v\rangle_H.$$

Since

$$|\langle J'(u), v\rangle_H| \leq \|J'(u)\| \quad \text{and} \quad \langle J'(u), -\frac{J'(u)}{\|J'(u)\|}\rangle = -\|J'(u)\|,$$

$d = -\frac{J'(u)}{\|J'(u)\|}$ is the steepest descent direction. Since we are looking for a critical point $u^* \in H$ such that $\|J'(u^*)\| = 0$, the normalization of the gradient will introduce an extra error in numerical computation and also when a stepsize s is used, s can absorb the length of a descent direction d, thus we may just call $d = -J'(u)$ the steepest descent direction.

Next let $\phi : H \to H$ be a continuous mapping and consider the composite function $J(\phi(\cdot)) : H \to \mathbb{R}$. It is clear that a vector $d \in H$ with $\langle J'(\phi(v)), d\rangle < 0$ is not necessarily a descent direction of $J(\phi(\cdot))$ at v. However, we have

Lemma 2.3. *Let p be local L-$\perp$ selection of J at $v \in S_{L^\perp}$. If p is continuous at v and $p(v) \notin L$, then any $d \in H$ with $d \perp \{L, v\}$ and $\langle J'(p(v)), d\rangle < 0$ is a descent direction of $J(p(\cdot))$ at v along a nonlinear variation*

$$v(s) = \frac{v + sd}{\sqrt{1 + s^2\|d\|^2}} \in S_{L^\perp}.$$

In particular, $d = -J'(p(v))$ is a descent direction at v and $d = -\frac{J'(u)}{\|J'(u)\|}$ is the steepest descent direction of $J(p(\cdot))$ along the nonlinear variation $v(s)$.

Proof. We have $p(v) \equiv t_v v + v_L$ for some $t_v \neq 0$ and $v_L \in L$ where $|t_v| = \mathrm{dis}(p(v), L)$. If $d \perp \{L, v\}$ with $d \neq 0$, we define a nonlinear variation at $v \in S_{L^\perp}$ by

$$v(s) = \frac{v + sd}{\sqrt{1 + s^2\|d\|^2}} \in S_{L^\perp} \tag{2.9}$$

where $s > 0$ if $t_v > 0$ and $s < 0$ if $t_v < 0$. It follows from $1 < 1 + \frac{s^2\|d\|^2}{(1+\sqrt{1+s^2\|d\|^2})^2} < 2$ and the orthogonality, we obtain

$$\frac{|s|\|d\|}{\sqrt{1 + s^2\|d\|^2}} < \|v(s) - v\| < \frac{\sqrt{2}|s|\|d\|}{\sqrt{1 + s^2\|d\|^2}}. \tag{2.10}$$

Thus we have

$$\begin{aligned} J(p(v(s))) - J(p(v)) &= \langle J'(p(v)), p(v(s)) - p(v)\rangle + o(\|p(v(s)) - p(v)\|) \\ &= \frac{t_s s}{\sqrt{1 + s^2\|d\|^2}}\langle J'(p(v)), d\rangle + o(\|p(v(s)) - p(v)\|), \end{aligned} \tag{2.11}$$

where since $v(s) \to v$ as $s \to 0$ and p is continuous at v, we have $\|p(v(s)) - p(v)\| \to 0$ and $t_s \to t_v$ as $s \to 0$. When $\langle J'(v), d\rangle < 0$ and $0 < \lambda < 1$, for $|s| > 0$ small we obtain

$$J(p(v(s))) - J(p(v)) < \lambda \frac{t_v s}{\sqrt{1 + s^2\|d\|^2}}\langle J'(p(v)), d\rangle \equiv \lambda t_v \langle J'(p(v)), v(s) - v\rangle < 0. \tag{2.12}$$

Thus d, in particular, $d = -J'(p(v))$, is a descent direction of $J(p(\cdot))$ at v along the nonlinear variation $v(s)$. Next we note that when s is small, t_s is close to t_v, the term $\frac{t_s s}{\sqrt{1+s^2\|d\|^2}}\langle J'(p(v)), d\rangle$ attains its minimum $-\frac{t_s s}{\sqrt{1+s^2}}\|J'(p(v))\|$ for all $d \in H$ with $\|d\| = 1$. Therefore $d = -J'(p(v))$ is the steepest descent direction of $J(p(\cdot))$ at v along the nonlinear variation $v(s)$. ∎

Lemma 2.3 shows one of the obvious advantages of our definition of a local L-$\perp$ selection. Once a descent direction is selected, we want to know how far it should go, in other words, we want to establish a stepsize rule.

Since $\sqrt{1+s^2\|d\|^2} \to 1$ as $s \to 0$, (2.12) can be rewritten as

$$J(p(v(s))) - J(p(v)) < \lambda t_v s \langle J'(p(v)), d\rangle. \tag{2.13}$$

When $d = -J'(p(v))$ is chosen and $|s| > 0$ is small, by using (2.10),

$$J(p(v(s))) - J(p(v)) < -\lambda t_v s \|J'(p(v))\|^2 \le -\frac{\lambda t_v}{\sqrt{2}}\|J'(p(v))\|\,\|v(s) - v\|.$$

The above analysis can be summarized as

Lemma 2.4. *Let H be a Hilbert space, L be a closed subspace of H and $J \in C^1(H,\mathbb{R})$. Let p be a local L-$\perp$ selection of J at a point $v \in S_{L^\perp}$ such that p is continuous at v and $p(v) \notin L$, then either $d \equiv -J'(p(v)) = 0$ or for each $0 < \lambda < 1$, there exists $s_0 > 0$ such that*

$$\begin{aligned} J(p(v(s))) - J(p(v)) &< -\lambda t_v s \|J'(p(v))\|^2 \\ &\le -\frac{\lambda |t_v|}{\sqrt{2}}\|J'(p(v))\|\|v(s) - v\|, \quad \forall\, 0 < |s| < s_0,\ t_v s > 0, \end{aligned} \tag{2.14}$$

where

$$p(v) = t_v v + v_L, \quad v_L \in L \quad \textit{and} \quad v(s) = \frac{v + sd}{\sqrt{1+s^2\|d\|^2}} \in S_{L^\perp},$$

$s > 0$ if $t_v > 0$ and $s < 0$ if $t_v < 0$. ∎

Lemma 2.4 has two outcomes. The first is a local characterization of a saddle point as stated in Lemma 2.5 and the second is that the inequality (2.14) can be used to define a stepsize rule in a numerical algorithm.

When $v \in S_{L^\perp}$ is a local minimum point of $J(p(\cdot))$ on $S_{L^\perp}$, if $p(v) \notin L$, inequality (2.14) will not be satisfied, then $u = p(v)$ must be a critical point. The condition $p(v) \notin L$ is important to ensure that the saddle point $u = p(v)$ is a new one outside L; if $p(v) \in L$, i.e., $u \equiv p(v) = v_L$ for some $v_L \in L$, u may fail to be a critical point or a new critical point. If $u \equiv p(v) = v_L$ is an isolated L-$\perp$ point in L, then $u \equiv p(v) = v_L$ is still a critical point. To see this, suppose $d \equiv -J'(p(v)) \neq 0$, let $v(s) = \frac{v+sd}{\sqrt{1+s^2\|d\|^2}}$, and consider $p(v(s)) = t_s v(s) + v_L(s)$. Since $p(v(s)) \to p(v) = v_L$ as $s \to 0$ and $\langle J'(p(v)), d\rangle < 0$, from

(2.11), if $t_s \neq 0$ for some $|s| > 0$ sufficiently small, we have $t_s s > 0$ and

$$J(p(v(s))) - J(p(v)) < \frac{t_s s}{2\sqrt{1 + s\|d\|^2}} \langle J'(p(v)), d \rangle < 0,$$

which implies that if v is a local minimum point of $J(p(\cdot))$, we must have $J'(p(v)) = 0$. Next if $t_s = 0$ i.e., $p(v(s)) \in \mathcal{N}(v_L) \cap L$ for all $s > 0$ sufficiently small. But v_L is an isolated L-$\perp$ point of J, we have $p(v(s)) = v_L(s) = v_L$. By the definition of p, $J'(p(v(s))) = J'(v_L)$ is orthogonal to $v(s)$. But $J'(v_L) \perp v$, it follows that $J'(v_L)$ is orthogonal to $d = -J'(v_L)$, i.e.,$J'(v_L) = 0$. We have established the following local characterization of a critical point.

Lemma 2.5. *Let H be a Hilbert space, L be a closed subspace of H and $J : H \to \mathbb{R}$ be a C^1 functional. Let $v \in S_{L^\perp}$ and p be a local L-$\perp$ selection of J at v such that*

(a) p is continuous at v,

(b) either $p(v) \notin L$ or $p(v) \in L$ is an isolated L-$\perp$ point of J,

(c) v is a local minimum point of the function $J(p(\cdot))$ on $S_{L^\perp}$,

then $u = p(v)$ is a critical point of J. ∎

Example 2.1. *First let us examine $f(t) = \frac{1}{2}t^2 - \frac{2}{3}|t|^3 + \frac{1}{4}t^4$. We have $f'(t) = t(1 - |t|)^2$. Thus $t = 0, \pm 1$ are three critical points of f. Since f' will not change sign near $t = \pm 1$, $t = \pm 1$ are two saddle points of f. While $f''(0) = 1$ implies that $t = 0$ is a local (global) minimum point. Next for each $x = (x_1, x_2) \in \mathbb{R}^2$, denote $\|x\|^2 = x_1^2 + x_2^2$ and define*

$$J(x) = \frac{\frac{1}{2}\|x\|^2 - \frac{2}{3}\|x\|^3 + \frac{1}{4}\|x\|^4}{\|\frac{x}{\|x\|} - (1,0)\|\|\frac{x}{\|x\|} - (0,1)\|}$$

and $J(0) = 0$. Then $J(x)$ is well-defined in $\mathbb{R}^2$ except along the lines $(x_1, 0)$ and $(0, x_2)$. It is clear that $x = (0,0)$ is a global minimum and a trivial critical point. Note that for $t > 0$

$$J(tx) = \frac{\frac{t^2}{2}\|x\|^2 - \frac{2t^3}{3}\|x\|^3 + \frac{t^4}{4}\|x\|^4}{\|\frac{x}{\|x\|} - (1,0)\|\|\frac{x}{\|x\|} - (0,1)\|}.$$

To find other critical points, let $L = \{0\}$. For each $\|x\| = 1$, $x \neq (1,0)$ or $(0,1)$ the local L-$\perp$ selection $p(x) = tx$ where t is solved from

$$0 = \frac{d}{dt}J(tx) = \frac{t - 2t^2 + t^3}{\|\frac{x}{\|x\|} - (1,0)\|\|\frac{x}{\|x\|} - (0,1)\|}.$$

It follows $t = 1$ or $p(x) = x$ is a saddle (not a local maximum) point of J in the direction of x. We have

$$J(p(x)) = \frac{1}{\|\frac{x}{\|x\|} - (1,0)\| \|\frac{x}{\|x\|} - (0,1)\|} \frac{1}{12}.$$

Thus

$$\min_{\|x\|=1} J(p(x)) \Longleftrightarrow \max_{0<\theta<\frac{\pi}{2}} \Big((\cos\theta - 1)^2 + \sin^2\theta \Big) \Big(\cos^2\theta + (\sin\theta - 1)^2 \Big).$$

By taking derivative, it leads to $\sin\theta(1 - \sin\theta) = \cos\theta(1 - \cos\theta)$, i.e., two local maxima are attained at $\theta = \frac{\pi}{4}$ and $\frac{5\pi}{4}$. Thus we conclude that $x = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ are two min-orthogonal saddle points of J, which cannot be characterized by a min-max method. Since $J(x) > 0$ for any $x \neq 0$ and $J(x) = +\infty$ for $x = (x_1, 0)$ or $(0, x_2)$, the function $J(x)$ has no mountain pass structure at all. Thus the wellknown mountain pass lemma, a minimax approach, cannot be applied.

Definition 2.3. *Let $v^* \in S_{L^\perp}$ and p be a local selection of P at v^*. A point $u^* = p(v^*)$ is said to be a U-turn critical point of J relative to L and p if $u^* \in L$ and there are $v \in L^\perp$, $v \neq 0$, a neighborhood $\mathcal{N}(u^*)$ of u^* and $s_0 > 0$ such that either*

$$(\mathcal{N}(u^*) \cap \{\{L, v^*\}, v\}) \cap p(v^*(s)) = \emptyset, \quad \forall 0 < s \leq s_0,$$

or

$$(\mathcal{N}(u^*) \cap \{\{L, v^*\}, -v\}) \cap p(v^*(s)) = \emptyset, \quad \forall 0 > s \geq -s_0,$$

where

$$v^*(s) = \frac{v^* + sv}{\|v^* + sv\|}.$$

In the above definition, we need $u^* = p(v^*) \in L$, for if $d(u^*, L) > 0$, u^* will not satisfy the condition.

Theorem 2.3. *Let $v^* \in S_{L^\perp}$ and p be a local L-$\perp$ selection of J w.r.t. L at v^* and continuous at v^*. Assume either $d(p(v^*) \notin L$ or $p(v^*) \in L$ is an isolated non U-turn critical point of J relative to L and p. If there is a neighborhood $\mathcal{N}(v^*)$ s.t. $v^* = \arg\min_{v \in \mathcal{N}(v^*) \cap S_{L^\perp}} J(p(v))$, then $u^* = p(v^*)$ is a critical point of J.*

Proof Suppose that $d = -J'(u^*) \neq 0$. There is $s_0 > 0$ such that for $0 < s < s_0$, we have

$$v^*(s) = \frac{v^* + sd}{\|v^* + sd\|} \in \mathcal{N}(v^*) \cap S_{L^\perp}, \quad \bar{v}^*(s) = \frac{v^* - sd}{\|v^* - sd\|} \in \mathcal{N}(v^*) \cap S_{L^\perp}.$$

If $d(p(v^*), L) > 0$, i.e., $u^* \equiv p(v^*) = t^*v^* + v_L^*$ for some $t^* \neq 0$ and $v_L^* \in L$,

$$p(v^*(s)) = t_s v^*(s) + v_L^*(s) \quad \text{and} \quad p(\bar{v}^*(s)) = \bar{t}_s \bar{v}^*(s) + \bar{v}_L^*(s) \tag{2.15}$$

for some scalars t_s, $\bar{t}_s$ and $v_L^*(s), \bar{v}_L^*(s)$ in L. By the continuity of p at v^*, when $s > 0$ is small, t_s and $\bar{t}_s$ have the same sign as that of t^*. It follows

$$\langle J'(u^*), p(v^*(s)) - p(v^*) \rangle = \frac{-t^s s}{\sqrt{1 + s^2\|d\|^2}} \|d\|^2, \tag{2.16}$$

and

$$\langle J'(u^*), p(\bar{v}^*(s)) - p(v^*) \rangle = \frac{+\bar{t}^s s}{\sqrt{1 + s^2\|d\|^2}} \|d\|^2. \tag{2.17}$$

The right hand sides in (2.16) and (2.17) are in opposite signs, so v^* can not be a local minimum point of $J(p(\cdot))$ in $\mathcal{N}(v^*) \cap S_{L^\perp}$ and it contradicts to our assumption.

Next if $u^* \equiv p(v^*) = v_L^* \in L$, there are two cases, either (a) for any $0 < \bar{s} < s_0$, there are $0 < s^1, s^2 < \bar{s}$ such that $t_{s^1} \neq 0$ and $\bar{t}_{s^2} \neq 0$ or (b) there is $0 < \bar{s} < s_0$ such that either $t_{s^1} = 0$ for all $0 < s^1 < \bar{s}$ or $\bar{t}_{s^2} = 0$ for all $0 < s^2 < \bar{s}$. In case of (a), since u^* is not a U-turn critical point of J relative to L and p, it implies that the right hand sides in (2.16) and (2.17) are in opposite sign. Thus it leads to the same contradiction. In case of (b), say $t_{s^1} = 0$ for all $0 < s^1 < \bar{s}$, that is $p(v^*(s)) = v_L^*(s) \in L \cap \mathcal{N}(u^*)$ due to the continuity and $J'(v_L^*(s)) \perp \{L, v^(s)\}$. But $u^* = v_L^*$ is the only critical point of J relative to L, we have $v_L^*(s) = v_L^*$. Thus $J'(u^*) \perp \{L, v^*, v^*(s)\}$, which implies that $J'(u^*) \perp d = -J'(u^*)$. It is impossible since we have supposed that $d = -J'(u^*) \neq 0$. ∎

Definition 2.4. *A function $J \in C^1(H)$ is said to satisfy the Palais-Smale (PS) condition, if any $\{u_n\} \in H$ with $J(u_n)$ bounded and $J'(u_n) \to 0$ has a convergent subsequence.*

The proof of the following existence result is similar to that of Theorem 1.2 in [11] and therefore omitted.

Theorem 2.4. *Assume that $J \in C^1(H, \mathbb{R})$ satisfies (PS) condition and $L \subset H$ is a closed subspace. If there exist an open set $O \subset L^\perp$ and a local L-$\perp$ selection p of J defined on $\bar{O} \cap S_{L^\perp}$ such that* **(a)** $c = \inf_{v \in O \cap S_{L^\perp}} J(p(v)) > -\infty$, **(b)** $J(p(v)) > c \quad \forall v \in \partial\bar{O} \cap S_{L^\perp}$, **(c)** *$p(v)$ is continuous on $\bar{O} \cap S_{L^\perp}$,* **(d)** *$d(p(v), L) \geq \alpha$ for some $\alpha > 0$ and all $v \in O \cap S_{L^\perp}$, then c is a critical value, i.e, there exists $v^* \in O \cap S_{L^\perp}$ such that*

$$J'(p(v^*)) = 0, \quad J(p(v^*)) = c = \min_{v \in O \cap S_{L^\perp}} J(p(v)). \qquad \blacksquare$$

Note that to apply the Ekeland's variational principle to J, J has to be bounded from below. However, in general, J is not bounded from below. Instead we assume that $J(p(v))$ is bounded from below. Then if for $L = \{0\}$, $J(p(v))$ is bounded from below, we can conclude that $J(p(v))$ is bounded from below for any closed subspace L of H. For example, the function J defined in (2.2) for the semilinear elliptic PDE (2.1) is bounded neither from below nor from above. However 0 is the only local minimum point of J with $J(0) = 0$. Along each direction v with $\|v\| = 1$ there is only one local maximum point $u = t^*v$ characterized by (2.4). Therefore we have $J(u) > 0$ for all such local maximum points u. Thus $\inf_{v \in S_{L^\perp}} J(p(v)) > 0$.

When we replace the local maximization in Steps 2 and 5 in the local minimax algorithm [11] by a local orthogonalization, we obtain a local min-$\perp$ algorithm. To be more specific, when a point $v \in S_{L^\perp}$ is given, to evaluate $p(v)$ as in Steps 2 and 5 in the flow chart, for the local minimax algorithm, we find a local maximum point $p(v)$ of J in $\{L, v\}$. Thus it is quite natural to expect that $p(v) \notin L$. For a local min-$\perp$ algorithm, we find a point $p(v) \in \{L, v\}$ such that $J'(p(v)) \perp \{L, v\}$ and $p(v) \notin L$. But L is spanned by previously found critical points, say, $L = \text{span}\{w_1, ..., w_n\}$. An L-$\perp$ point $p(v) = t_0 v + t_1 w_1 + \cdots + t_n w_n$ is solved from the system

$$\begin{aligned}
\langle J'(t_0 v + t_1 w_1 + ... + t_n w_n), v\rangle &= 0,\\
\langle J'(t_0 v + t_1 w_1 + ... + t_n w_n), w_1\rangle &= 0,\\
&\cdots\\
\langle J'(t_0 v + t_1 w_1 + ... + t_n w_n), w_n\rangle &= 0
\end{aligned}$$

for $t_0, t_1, ..., t_n$. Each one w_i in L will trivially satisfy the system. Since $p(v) \notin L$ is important in our theoretical setting for finding a new solution, those choices must be excluded for $p(v)$. Once this condition is satisfied in implementation, convergence results of the local min-$\perp$ algorithm similar to those in [11] can be established almost identically.

A Numerical Local Min-Orthogonal Algorithm

Step 1: Given $\varepsilon > 0, \lambda > 0$ and n previously found critical points $w_1, w_2, \ldots, w_n$ of J, of which w_n has the highest critical value. Set $L = \text{span}\{w_1, w_2, \ldots, w_n\}$. Let $v^1 \in S_{L^\perp}$ be an ascent direction at w_n. Let $t_0^0 = 1$, $v_L^0 = w_n$ and set $k = 0$;

Step 2: Using the initial guess $u = t_0^k v^k + v_L^k$, solve for $u^k \equiv p(v^k) \in \{L, v^k\} \setminus L$ from

$$\langle J'(u^k), v^k \rangle = 0, \quad \langle J'(u^k), w^j \rangle = 0, j = 1., , n.$$

Denote $t_0^k v^k + v_L^k = u^k \equiv p(v^k)$;

Step 3: Compute the steepest descent vector $d^k = -J'(u^k)$;

Step 4: If $\|d^k\| \leq \varepsilon$ then output $w_{n+1} = u^k$, stop; else goto Step 5;

Step 5: Set $v^k(s) = \dfrac{v^k + sd^k}{\|v^k + sd^k\|}$ and find

$$s^k = \max\left\{\frac{\lambda}{2^m} \;\middle|\; m \in \mathbb{N}, 2^m > \|d^k\|, J(p(v^k(\frac{\lambda}{2^m}))) - J(w^k) \leq -\frac{t_0^k}{2}\|d^k\| \, \|v^k(\frac{\lambda}{2^m}) - v^k\|\right\}.$$

Initial guess $u = t_0^k v^k(\frac{\lambda}{2^m}) + v_L^k$ is used to find $p(v^k(\frac{\lambda}{2^m}))$ in $\{L, v^k(\frac{\lambda}{2^m})\} \setminus L$ as similar in **Step 2** and where t_0^k and v_L^k are found in **Step 2**.

Step 6: Set $v^{k+1} = v^k(s^k)$ and update $k = k + 1$ then goto **Step 2**. ∎

It is wellknown that a steepest descent method may approximate an inflection point, not necessarily a local minimum point. But convergence results in [11] show that any limit point of the sequence generated by the algorithm is a critical point, not necessarily a min-$\perp$ saddle point. Local characterization of critical points presented in [10,11] cannot cover such cases. Now with the necessary and sufficient condition in the local characterization of critical points established in Theorem 2.1, it becomes clear that for a steepest descent method stops at a limit point, the orthogonal condition (2.5) must be satisfied. Thus it has to be a critical point.

3 Differentiability of an L-$\perp$ selection p

Continuity and/or differentiability condition of a peak selection p have been used to establish the local minimax theorem [10], to prove convergence of the local minimax algorithm [11 and to study local instability of minimax solutions in [24]. Since a peak selection p is defined by a local maximization process not an explicit formula, it is very difficult to check those conditions. When a peak selection is generalized to an L-$\perp$ selection p, various implicit function theorems can be used to check continuity or smoothness of p at certain point. For example, let us use the classical implicit function theorem to directly check if a local L-$\perp$ selection p is differentiable or not. Let $L = \{w_1, w_2, .., w_n\}$ and $v \in S_{L^\perp}$. Following the definition of p, $u^* = t_0 v + t_1 w_1 + ... + t_n w_n = p(v)$ is solved from (n+1) orthogonal conditions

$$\begin{aligned} F_0(v, t_0, t_1, ..., t_n) &\equiv \langle J'(t_0 v + t_1 w_1 + ... + t_n w_n), v\rangle = 0, \\ F_j(v, t_0, t_1, ..., t_n) &\equiv \langle J'(t_0 v + t_1 w_1 + ... + t_n w_n), w_j\rangle = 0, \quad j = 1, ..., n \end{aligned}$$

for $t_0, t_1, ..., t_n$. We have

$$\begin{aligned} \frac{\partial F_0}{\partial t_0} &= \langle J''(t_0 v + t_1 w_1 + ... + t_n w_n)v, v\rangle, \\ \frac{\partial F_0}{\partial t_i} &= \langle J''(t_0 v + t_1 w_1 + ... + t_n w_n)w_i, v\rangle, \quad i = 1, 2, ..., n. \\ \frac{\partial F_j}{\partial t_0} &= \langle J''(t_0 v + t_1 w_1 + ... + t_n w_n)v, w_j\rangle, \quad j = 1, 2, ..., n. \\ \frac{\partial F_j}{\partial t_i} &= \langle J''(t_0 v + t_1 w_1 + ... + t_n w_n)w_i, w_j\rangle, \quad i, j = 1, 2, ..., n. \end{aligned}$$

By the implicit function theorem, if the $(n+1) \times (n+1)$ matrix

$$Q'' = \begin{bmatrix} \langle J''(u^*)v, v\rangle, & \langle J''(u^*)w_1, v\rangle, & ... & \langle J''(u^*)w_n, v\rangle \\ \langle J''(u^*)v, w_1\rangle, & \langle J''(u^*)w_1, w_1\rangle, & ... & \langle J''(u^*)w_n, w_1\rangle \\ ... & & ... & \\ \langle J''(u^*)v, w_n\rangle, & \langle J''(u^*)w_1, w_n\rangle, & ... & \langle J''(u^*)w_n, w_n\rangle \end{bmatrix} \tag{3.18}$$

is invertible or $|Q''| \neq 0$, where $u^* = t_0 v + t_1 w_1 + ... + t_n w_n = p(v)$, then p is differentiable at and near v. This condition can be easily and numerically checked.

EXAMPLE 3.1. *Let us consider a functional of the form*

$$J(u) = \frac{1}{2}\langle Au, u\rangle - \int_\Omega F(u(x))\, dx$$

for $u \in H = H_0^1(\Omega)$ when Ω is a bounded open set in $\mathbb{R}^n$, $A : H \to H$ be a bounded linear self-adjoint operator and where $F'(t) = f(t)$ satisfies some standard growth and regularity conditions. We have

$$\langle J''(w)u, v\rangle = \langle Au, v\rangle - \int_\Omega f'(w(x))u(x)v(x)\,dx.$$

The condition that $\frac{f(t)}{t}$ is monotone has been used in the literature [17] to prove the existence of multiple solutions. Here we show that this condition implies that the L-$\perp$ selection p is the unique peak selection w.r.t. $L = \{0\}$ and p is differentiable at every $u \in H$ with $\|u\| = 1$.

First we note that $\frac{f(t)}{t}$ is monotone $\Longleftrightarrow$ either $f'(t)t - f(t) < 0 \quad \forall t \neq 0$ or $f'(t)t - f(t) > 0 \quad \forall t \neq 0$. For each $u \in S_{L^\perp}$,

$$J(tu) = \frac{t^2}{2}\langle Au, u\rangle - \int_\Omega F(tu(x))\,dx.$$

$$\frac{d}{dt}J(tu) = 0 \Longleftrightarrow \langle Au, u\rangle - \int_\Omega \frac{f(tu(x))}{tu(x)}u^2(x)\,dx = 0.$$

Since $\frac{f(t)}{t}$ is monotone, for each such u, there exists at most one t_u such that $\frac{d}{dt}J(tu)|_{t=t_u} = 0$. Thus the L-$\perp$ selection is unique $p(u) = t_u u$. Furthermore, if $\frac{d}{dt}J(t_u u) = 0$, we have

$$\begin{aligned}\langle J''(t_u u)u, u\rangle &= \langle Au, u\rangle - \int_\Omega f'(t_u u(x))u^2(x)\,dx \\ &= -\int_\Omega \left(f'(t_u u(x)) - \frac{f(t_u u(x))}{t_u u(x)}\right) u^2(x)\,dx \neq 0.\end{aligned}$$

Thus by the implicit function theorem, p is differentiable at u.

When $\dim(L) = n$, the differentiability of p can be computationally checked through verifying $|Q''| \neq 0$. This inequality has been numerically checked to be satisfied for all multiple solutions to superlinear elliptic equations numerically computed in [10,11], in particular, when a concentric annular domain is used, a rotation of a solution for any angle is still a solution. So each solution belongs to a one-parameter family of solutions and therefore a degenerate critical point.

This approach also plays an important role in computational local instability analysis of saddle points as studied in [24].

Remark 3.1. There are several advantages to use a local min-L-$\perp$ approach. The first, if we use a local min-max approach in numerical computation, theoretically we can embed a local min-max approach into a local min-L-$\perp$ approach, i.e., we can embed the graph of a peak mapping into the graph of a corresponding L-$\perp$ mapping. Thus any limit of the graph of a peak mapping is always in the graph of a corresponding L-$\perp$ mapping. The second, to solve for u in $\{L, v\}$ such that $J'(u) \perp \{L, v\}$ is equivalent to solving system of equations, it is much easier to determine the continuity or differentiability of a local L-$\perp$ selection than that of a local peak selection. For example, we may use various implicit function theorems to determine the continuity or differentiability of a local L-$\perp$ selection p near a point v. The disadvantages of using min-L-$\perp$ method are that we lost trace of instability index of a solution, since the solution found by the local min-L-$\perp$ method can be too general, e.g., the monkey saddles, to define a local instability index, and it is not easy to satisfy the condition $p(v) \notin L$, an important condition in out theoretical setting. As for a minimax saddle point, we can combine the merits of two approaches, i.e., use a local peak selection, or, a local maximum point in $\{L, v\}$ at the first level of the algorithm and treat it as a local L-$\perp$ selection to check its continuity or differentiability and to do other theoretical analysis.

Then an interesting question can be asked, when a local L-$\perp$ selection p is used to find a critical point $u^* = p(v^*)$ that happens to be a local maximum point of J in $\{L, v^*\}$, will such a local L-$\perp$ selection become a local peak selection near v^*? Theorem 2.6 in [24] positively answers the question.

The notion of an L-$\perp$ selection has been recently applied in [23] to define a modified pseudo gradient (flow) of a function, with which we are able to develop a local minimax method for finding multiple saddle points in Banach spaces, such as multiple solutions to a quasilinear elliptic PDE and eigenpairs to the p-Laplacian operators.

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