1. (10) Let a bond have yield to maturity y, and coupon rate R. Suppose the bond matures in $M/2$ years. Define the Macaulay duration of this bond.

   Let F be the face value of the bond. For each k between 1 and M, define the following terms:

$$A_k = \begin{cases} \dfrac{RF}{200} & \text{if} \quad 1 \le k < M \\ \left(1 + \dfrac{R}{200}\right) F & \text{if} \quad k = M \end{cases}$$

$$\omega_k = \frac{1}{P} A_k \left(1 + \frac{y}{200}\right)^{-k},$$

   where P is the price of the bond.

   The Macaulay duration is then defined as

$$D = \sum_{k=1}^{M} k\omega_k$$

2. (20) A bond matures in one year. Its yield to maturity is 4%, its coupon rate is 3%, and its face value is \$100.

   (a) Determine the arbitrage free price of the bond.

   Using the HP12C, with 4 in the interest register and 3 in the payment register, and dates Jan. 1, 2000 to Jan. 1, 2001 we get for the arbitrage free price of the bond

$$P_0 = 99.03$$

   (b) Determine the bonds Macaulay duration and its modified Macaulay duration.

$$\omega_1 = \frac{1}{99.03}(1.5)\left(1 + \frac{4}{200}\right)^{-1} = 0.0148$$

$$\omega_2 = 1 - \omega_1 = 1 - 0.0148 = 0.9852$$

$$D = \omega_1 + 2\omega_2 = 0.0148 + 2(0.9852) = 1.985$$

$$\hat{D} = \frac{D}{204} = \frac{1.985}{204} = 0.00973$$

   (c) Determine the convexity of the bond.

$$C = \frac{2\omega_1 + 6\omega_2}{(204)^2} = \frac{2(0.0148) + 6(0.9852)}{(204)^2} = 0.000143$$

3. (20) A treasury note matures in 5 years. Its current yield and coupon rate are 4 and 5.5 percent respectively. Anticipating that interest rates are about to rise you decide to short sell one of these bonds, face value of \$100. After one interest period you close out your position. Assume that your broker will invest any monies accrued to your account at an annual rate of 3%, and that there is no change in the bond's yield to maturity. Determine the net profit or loss.

   Since the note is short sold your account will be credited with the selling price of the note and this amount will increase by 1.5% over the period your position is open. When the position is closed your account will have one coupon payment deducted and the current price of the note will also be deducted when you buy back (close out your position) the note.

   The current price of the note is $P_0 = 106.74$. The price of the note 6 months later, assuming there is no change in its yield to maturity, is $P_1 = 106.12$. After 6 months the amount P_0 invested at 3% grows to

$$106.74\left(1+\frac{3}{200}\right) = 108.34$$

   Thus, the net profit or loss equals

$$\begin{aligned} 108.34 - \frac{RF}{200} - P_1 &= 108.34 - 2.75 - 106.12 \\ &= -0.53 \end{aligned}$$

Data for three bonds are given below: M denotes number of interest periods until the bond matures, y is the yield to maturity, R is the coupon rate, P is the price, D is the Macaulay duration, and C is the bond's convexity. Each bond has a face value of \$100. Use this information to answer questions 4 , 5, and 6.

Bond	M	y	R	P	D	C
1	6	7	0	81.35	6	0.00098
2	14	4.5	4	97.02	12.318	0.00423
3	20	5	4	92.21	16.511	0.00771

4. (10) A portfolio consisting of the above three bonds is established in which 100 of the first bond, 250 of the second bond, and −200 of the third bond are purchased.

(a) What is the duration of this portfolio?

Let Π_0 denote the value of the portfolio. Then the duration of the portfolio equals

$$\begin{aligned} D_\Pi &= \frac{1}{\Pi_0}\left(x_1 P_1 \frac{D_1}{200+y_1} + x_2 P_2 \frac{D_2}{200+y_2} + x_3 P_3 \frac{D_3}{200+y_3}\right), \text{ where} \\ \Pi_0 &= x_1 P_1 + x_2 P_2 + x_3 P_3 = 100\,(81.35) + 250\,(97.02) - 200\,(92.21) \\ &= 13,948.00\,. \end{aligned}$$

Thus,

$$\begin{aligned} D &= \frac{1}{13,948}\left(100\,(81.35)\,\frac{6}{207} + 250\,(97.02)\,\frac{12.318}{204.5} - 200\,(92.21)\,\frac{16.511}{205}\right) \\ &= \frac{1}{13,948}\,(211.44454) = 0.01516\ . \end{aligned}$$

(b) What is the convexity of this portfolio?

The convexity of the portfolio equals

$$C_\Pi = \frac{1}{\Pi_0}\,(x_1 P_1 C_1 + x_2 P_2 C_2 + x_3 P_3 C_3)$$

$$= \frac{1}{13,948}\,(100\,(81.35)\,(0.00098) + 250\,(97.02)\,(0.00423) - 200\,(92.21)\,(0.00771))$$

$$= \frac{1}{13,948}\,(-31.616\,87) = -0.002267$$

5. (20) Let x_1, x_2, and x_3 denote the amounts of the three bonds contained in a portfolio, with x_i being the amount of bond i. Suppose $x_1 = 100$, and $x_2 = 100$. If the portfolio is δ hedged what does x_3 equal?

The portfolio is δ hedged when its duration equals zero. This leads to the equations

$$\begin{aligned}
\frac{1}{\Pi_0}\left(x_1 P_1 \frac{D_1}{200+y_1} + x_2 P_2 \frac{D_2}{200+y_2} + x_3 P_3 \frac{D_3}{200+y_3}\right) &= 0 \\
\left(x_1 P_1 \frac{D_1}{200+y_1} + x_2 P_2 \frac{D_2}{200+y_2} + x_3 P_3 \frac{D_3}{200+y_3}\right) &= 0 \\
100\,(81.35)\,\frac{6}{207} + 100\,(97.02)\,\frac{12.318}{204.5} + x_3\,(92.21)\,\frac{16.511}{205} &= 0 \\
820.194\,34 + 7.426728\,x_3 &= 0
\end{aligned}$$

The solution to the last equation is

$$x_3 = \frac{-820.194\,34}{7.426728} = -110.438$$

6. (20) Suppose there is an instantaneous increase in the yield of 15 basis points for the second bond.

   (a) Use the above data to calculate a second order approximation to the new price of the second bond.

$$
\begin{aligned}
P(4.5+0.15) &\approx P(4.5) + P'(4.5)(0.15) + \frac{P''(4.5)}{2}(0.15)^2 \\
&= P(4.5) - P(4.5)\hat{D}(4.5)(0.15) + \frac{P(4.5)\,C(4.5)}{2}(0.15)^2 \\
&= 97.02\left(1 - \frac{12.318}{204.5}(0.15) + \frac{0.00423}{2}(0.15)^2\right) \\
&= 97.02\,(0.991) \\
&= 96.15
\end{aligned}
$$

   (b) Use the above data to calculate a first order approximation to the Macaulay duration of the second bond.

   One way to do this is to realize that approximating the duration is equivalent to approximating the first derivative. The relevant equations are:

$$P'(4.5+0.15) \approx P'(4.5) + P''(4.5)(0.15)$$

   In terms of duration and convexity the above can be writen as

$$
\begin{aligned}
-P(4.65)\frac{D(4.65)}{204.65} &\approx -P(4.5)\frac{D(4.5)}{204.5} + P(4.5)\,C(4.5)(0.15) \\
-\frac{96.15}{204.65}D(4.65) &\approx -(97.02)\frac{12.318}{204.5} + 97.02\,(0.00423)(0.15) \\
&= -5.7824 \quad \text{Thus,} \\
D(4.65) &\approx 5.7824\frac{204.65}{96.15} = 12.307
\end{aligned}
$$