

Assignment # 4

(Due date: Friday, Oct. 29, 1999)

In the first part of this assignment, you will go through a directed study aimed at completing the study of countability and separation axioms. In the first portion all problems are classified as additional problems. You do not need to hand these in.

1 Remarks on countability and separation

Recall the following examples

- a. **The Sierpinski space \mathcal{S} :** This is the space $\mathcal{S} = \{a, b\}$ consisting of two points with the topology $\mathcal{T} = \{\mathcal{S}, \emptyset, \{a\}\}$.
- b. \mathbb{R}_{fin} : The reals with the finite complement topology = Zariski topology on \mathbb{R} .
- c. **The Sorgenfrey line \mathbb{R}_ℓ :** This is another name for the reals with the lower limit topology.
- d. **The Sorgenfrey plane \mathbb{R}_ℓ^2 :** This is the plane \mathbb{R}^2 with the product topology $\mathbb{R}_\ell \times \mathbb{R}_\ell$, i.e. the Sorgenfrey line times itself.
- e. $(I \times I)_{lex}$: This is the unit square with the lexicographic order topology.
- f. S_Ω : The first uncountable ordinal, with the order topology.
- g. \overline{S}_Ω : $\overline{S}_\Omega = S_\Omega \cup \{\Omega\}$ is the compactified first uncountable ordinal, with the order topology where Ω is the maximal element.
- h. **The slotted plane \mathbb{R}_{sl}^2 :** This is the plane \mathbb{R}^2 with the topology given in Exam I, part 1, Problem 5. Here, the basic open sets around a point \underline{a} have the form $(B_\epsilon(\underline{a}) - \{L_1 \cup \dots \cup L_k\}) \cup \{\underline{a}\}$, where $B_\epsilon(\underline{a})$ are ϵ -balls around \underline{a} and the L_i 's are lines through \underline{a} .
- i. \mathbb{R}_{zar}^n : \mathbb{R}^n with the Zariski topology.

The following facts are known to you.

Facts 1.

1. Compact metric spaces and separable metric spaces are *2nd* countable. (Shown in class.)
2. \mathbb{R}_{fin} is Lindelöf and separable, but not *1st* countable. (Shown in class. This also holds for \mathbb{R}_{zar}^2 .)
3. \mathbb{R}_ℓ is *1st* countable, Lindelöf and separable, but not *2nd* countable. (Shown in class.)
4. It is easy to see that $(I \times I)_{lex}$ is *1st* countable. Since it is compact, then it is Lindelöf, however it is not separable.
5. The slotted plane \mathbb{R}_{sl}^2 is separable, but it is not *1st* countable and it is not Lindelöf. (You have proven this in Exam I, Part I, problem 5.)

Additional Problem 1. Verify that the space S_Ω is first countable. (This was indicated in class.) Then show that it is neither Lindelöf nor separable.

Additional Problem 2. Since $\overline{S_\Omega}$ is compact, then it is Lindelöf. Show that it is neither *1st* countable nor separable.

Additional Problem 3. Consider the space $\mathbb{R}_\ell^2 = \mathbb{R}_\ell \times \mathbb{R}_\ell$.

a Show that \mathbb{R}_ℓ^2 is *1st* countable.

b Show that it is separable.

c Show that it is not Lindelöf, using the following steps. Show that the “anti-diagonal” line $L = \{(x, -x) \mid x \in \mathbb{R}\}$ is closed in \mathbb{R}_ℓ^2 . For each point $(a, -a)$ in L , define $V_a = [a, +\infty) \times [-a, +\infty)$. Consider the collection $\{\mathbb{R}_\ell^2 - L\} \cup \{V_a \mid a \in \mathbb{R}\}$ and draw some conclusions.

Additional Problem 4. Show that the space $\mathbb{R}_{fin} \times \mathbb{R}_\ell^2$ is separable, but is is neither *1st* countable nor Lindelöf.

The following facts are known to you.

Facts 2.

1. The Sierpinski space \mathcal{S} is T_0 but not T_1 . (In class.)
2. \mathbb{R}_{fin} and \mathbb{R}_{zar} are T_1 but not T_2 . (in class.)
3. \mathbb{R}_{sl}^2 is T_2 but not T_3 .
4. Metric spaces are T_4 .

The following facts have somewhat elaborate proofs. (See Munkres.)

Facts 3.

1. Every well-ordered set X is normal in the order topology.
2. The product $S_\Omega \times \overline{S}_\Omega$ is NOT normal. Since products of regular spaces are regular (why?), one gets an example of a regular space which is not normal.
3. The space \mathbb{R}_ℓ is normal, but \mathbb{R}_ℓ^2 is not. This is another example of a regular space which is not normal.

USE the examples above to draw the following conclusions:

- Products of normal spaces do not need to be normal.
- Subspaces of normal spaces are not necessarily normal.

Additional Problem 5. a Let $\{X_\alpha | \alpha \in I\}$ and $\{Y_\beta | \beta \in J\}$ be two families of spaces and let $\phi : I \rightarrow J$ be a bijection. If for each $\alpha \in I$, one has a homeomorphism $X_\alpha \cong Y_{\phi(\alpha)}$, show that the product $\prod_\alpha X_\alpha$ is homeomorphic to $\prod_\beta Y_\beta$.

b Let X be a topological space, and let Y denote the cartesian product $X^{\mathbb{Z}^+}$. Use the previous item to show that Y is homeomorphic to any finite product $Y \times \dots \times Y$.

2 The Problems

You must hand these in.

Problem 1. Let X be the discrete subset $\{0, 2\}$ of \mathbb{Z} , and let Y be the countable product $Y = \prod_{i \in \mathbb{Z}_+} X$ of X with itself.

1. Show that Y is compact, Hausdorff but not discrete.
2. Show that Y is *totally disconnected*, in other words, that all its connected components consist of a single point.
3. Denote an element of X by $\underline{a} = (a_1, a_2, \dots)$ and define a map $f : Y \rightarrow \mathbb{R}$ by $f(\underline{a}) = \sum_{i=1}^{\infty} \frac{a_i}{3^i}$.
 - (a) Show that f is continuous;
 - (b) Show that f is one-to-one and that it gives a homeomorphism between X and its image.
 - (c) (EXTRA) Show that the image of f is the Cantor set. Therefore, X is homeomorphic to the Cantor set.

Problem 2. a Show that the product of completely regular spaces is completely regular.

b Show that a subspace of a completely regular spaces is completely regular. Explain why this shows that $S_{\Omega} \times \overline{S}_{\Omega}$ is completely regular. (But not normal, cf. Facts above.)

Problem 3. a Show that a connected normal space having more than one point is uncountable.

b Show that a connected regular space having more than one point is uncountable.

Problem 4. Show that every locally compact Hausdorff space is regular.

Problem 5. Solve Problem 5, p. 24, Bredon's book. (Then observe that it can immediately be used to solve Problem 6.)