Problem 1 (18%). Denote \( Y = S^1 \times \cdots \times S^1 \) and let \( X \) be the product \( \mathbb{R}P^3 \times S^2 \times L(7; q_1, q_2, q_3) \), where \( L(7; q_1, q_2, q_3) \cong S^5/(\mathbb{Z}/7\mathbb{Z}) \) is a lens space.

a: Compute the fundamental group of \( X \), clearly exhibiting the results that you used.

b: Exhibit a simply-connected covering space for \( Y \).

c: Show that any map \( f : X \to Y \) is homotopic to a constant map.

Problem 2 (16%). Let \( p : X \to Y \) be a covering map sending \( x_0 \in X \) to \( y_0 \in Y \). Show that if \( p_\# : \pi_1(X, x_0) \to \pi_1(Y, y_0) \) is an isomorphism then \( p \) is a homeomorphism.

Problem 3 (16%). Compute the fundamental group of the following spaces:

a: \( X = \mathbb{R}^5 - L \), where \( L \) is a 3-dimensional linear subspace of \( \mathbb{R}^5 \).

b: \( X = S^2 \cup J \), where \( S^2 \) is the unit sphere in \( \mathbb{R}^3 \) and \( J \) is the set of all points in the three coordinate axes which lie inside the closed unit ball. In other words,

\[
J = ([-1, 1] \times \{0\} \times \{0\}) \cup ([0] \times [-1, 1] \times \{0\}) \cup ([0] \times \{0\} \times [-1, 1]).
\]

Problem 4 (16%). Show that every smooth manifold \( M \) has a universal covering space \( \tilde{M} \to M \). Assuming that \( \tilde{M} \) is second countable show that it can be given a differentiable structure such that \( f \) is smooth and that the group of deck transformations \( \Delta \) acts on \( \tilde{M} \) via diffeomorphisms.

Problem 5 (18%). Let \( G \) be a topological group and suppose that \( G \) has a universal covering space \( \tilde{G} \) (as a topological space). Show that one can define a group structure on \( \tilde{G} \) making it into a topological group and such that the projection \( p : \tilde{G} \to G \) is a group homomorphism. Explain why this group structure on \( \tilde{G} \) is essentially unique.

(HINT: After fixing suitable base points, use lifting results.)

Problem 6 (16%). Describe all (up to equivalence) connected covering spaces of \( \mathbb{R}P^5 \times S^1 \times \mathbb{C}^* \), where \( \mathbb{R}P^5 \) is the 5-dimensional real projective space, \( S^1 \) is the unit circle, and \( \mathbb{C}^* \) is the the complex plane \( \mathbb{C} \) minus the origin.

REMARK: The solution should be very concise. Be very explicit about which equivalence you are using.