Problem List I

Problem 1. A space $X$ is said to be contractible if the identity map $i_X: X \to X$ is nullhomotopic.

a. Show that any convex subset of $\mathbb{R}^n$ is contractible.

b. Show that a contractible space is path connected.

c. Show that if $Y$ is contractible, then for any $X$ the set $[X, Y]$ of homotopy classes of maps from $X$ to $Y$ has a single element.

d. Show that if $X$ is contractible and $Y$ is path connected, then $[X, Y]$ has a single element. Explain what may happen when $Y$ is not path connected.

Problem 2. Let $p: M \to N$ be a smooth surjective map between compact smooth manifolds of same dimension (i.e. $\dim M = \dim N$). Suppose that every point of $M$ is a regular point.

i. Show that every $y \in N$ has an open, path-connected neighborhood $U$ such that $p^{-1}(U)$ is a disjoint union $U_1 \sqcup \cdots \sqcup U_r$ of open subsets of $M$ having the property that the restriction $p|_{U_i}$ is a diffeomorphism between $U_i$ and $U$, for all $i = 1, \ldots, r$.

ii. Show that if $Y$ is connected, then the number $r$ of connected components of $p^{-1}(U)$ does not depend on $y$.

Problem 3. Solve the following problems:

i. Verify that the map $p_n: S^1 \to S^1$ defined by $p(z) = z^n$ satisfies the hypothesis in the previous question.

ii. Verify that the map $\rho: S^n \to \mathbb{R}P^n$, sending $x \in S^n$ to the line $\overline{0x}$ through the origin $0$ and $x \in \mathbb{R}^{n+1}$, also satisfies the hypothesis of previous question.

iii. Consider the continuous group homomorphism $p: \mathbb{R} \to S^1$ sending $x \in \mathbb{R}$ to $\exp 2\pi i x$. Show that for every proper connected subspace $U \subset S^1$, the map $p$ carries every component of $p^{-1}(U)$ homeomorphically onto $U$.

Problem 4. Let $A \subset X$ be a retract of $X$ (i.e. there is $r: X \to A$ such that $r|_A = 1_A$) and pick $a_0 \in A$. Show that for all $n \geq 1$, the $n$-th homotopy group $\pi_n(A, a_0)$ is a subgroup of $\pi_n(X, a_0)$.
Problem 5. Consider the torus $T = (I \times I)/\sim$ obtained by identifying points $(x,0) \sim (x,1)$ and $(0,y) \sim (1,y)$, for all $x,y \in I$. The line segments $\{1\} \times I$ and $I \times \{1\}$ are both mapped into circles $C_1$ and $C_2$, respectively, under the quotient map $\rho: I \times I \to T$. Show that both $C_1$ and $C_2$ are retracts of $T$, but neither is a deformation retract of $T$.

Problem 6. Solve the following problems:

i. Let $f: s^{n-1} \to B$ be a map, and define $X = B \cup_f D^n$ be the space obtained by attaching an $n$-cell $D^n$ to $B$ via $f$. (The pair $(X,B)$ is called a relative cell complex.) Show that $B \subset X$ is a strong deformation retract of the space obtained from $X$ by deleting an interior point of $D^n$.

ii. Consider $X = S^m \times S^n$ and let $A \subset X$ be the subspace $A = S^m \vee S^n := \{s_0\} \times S^n \cup S^m \times \{t_0\}$. Show that $A$ is a strong deformation retract of the space obtained from $X$ by removing a point not in $A$.

Problem 7. Given a path $\gamma: I \to X$ from $x_0$ to $x_1$, let $h_\gamma: \pi_1(X,x_1) \to \pi_1(X,x_0)$ denote the induced isomorphism. Show that $\pi_1(X,x_0)$ is abelian if and only if for every pair $\alpha$ and $\beta$ of paths from $x_0$ to $x_1$ one has $h_\alpha = h_\beta$.

Problem 8. Answer the following questions:

a. Let $A \subset X$; suppose $r: X \to A$ is a retraction (i.e. $r(a) = a$ for all $a \in A$). If $a_0 \in A$, show that $r_\#: \pi_1(X,a_0) \to \pi_1(A,a_0)$ is surjective.

b. Let $L \subset \mathbb{R}^n$ be a 2-dimensional linear subspace of $\mathbb{R}^n$ and let $p \in L$ be an arbitrary point in $L$. Show that there is no retraction $r: \mathbb{R}^n \to L - \{p\}$.

Problem 9. Let $G$ be a topological group with operation $\cdot$ and identity element $e$. Let $\Omega(G,e)$ be the set of all loops in $G$ based at $e$. If $f, g \in \Omega(G,e)$, define a loop $f \bullet g$ by the rule:

$$(f \bullet g)(t) = f(t) \cdot g(t).$$

a. Show that $\Omega(G,e)$ is a group under the operation $\bullet$.

b. Show that this operation induces a group operation $\bullet$ on the set $\pi_1(G,e)$ of based homotopy classes of maps from $(S^1,1)$ to $(G,e)$.

c. Show that the group operation $\bullet$ on $\pi_1(G,e)$ coincides with the usual operation on the fundamental group (induced by the concatenation of paths).

(HINT: Compute $(f * C_e) \bullet (C_e * g)$.)
d. Show that for every point $x_0 \in G$, the fundamental group $\pi_1(G, x_0)$ is abelian.

**Problem 10.** Let $p_1: X_1 \to Y_1$ and $p_2: X_2 \to Y_2$ be covering maps. Show that $p_1 \times p_2: X_1 \times X_2 \to Y_1 \times Y_2$ is a covering map.

**Problem 11.** Let $p: X \to Y$ be a covering map ($Y$ connected). Show that if the fiber $p^{-1}(y_0)$ has $k$ elements for some $y_0 \in Y$, then $p^{-1}(y)$ has $k$ elements for all $y \in Y$.

**Problem 12.** Let $p: X \to Y$ and $q: Y \to Z$ be covering maps and let $\rho: X \to Z$ be $\rho := q \circ p$. Show that if $q^{-1}(z)$ is finite for all $z \in Z$ then $\rho$ is a covering map. Give a counterexample when this finiteness condition is not satisfied.

**Problem 13.** Let $p: X \to Y$ be a covering map.

a. If $Y$ is either regular, completely regular or locally compact Hausdorff, then so is $X$.

b. If $Y$ is compact and $p^{-1}(y)$ is finite for all $y \in Y$, then $X$ is compact.

**Problem 14.** Let $p: \mathbb{R} \to S^1$ denote the “exponential” covering map and let $p \times p: \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ be the product covering map. Let $f(t) = (\cos 2\pi t, \sin 2\pi t) \times (\cos 4\pi t \times \sin 4\pi t)$.

a. Sketch what $f$ looks like when $S^1 \times S^1$ is identified with the doughnut surface.

b. Find a lifting $\hat{f}$ of $f$ to $\mathbb{R} \times \mathbb{R}$.

**Problem 15.** Let $p: X \to Y$ be a covering map ($X$ is path connected). Show that if $B$ is simply connected, then $p$ is a homeomorphism.

**Problem 16.** Use the fact (proven in class) that the $n$-th homotopy group of the sphere $S^n$ is NOT trivial to give a proof that the sphere $S^n$ is not a retract of the closed ball $B^{n+1} \subset \mathbb{R}^{n+1}$.

**Problem 17.** Show that every vector field on $B^{n+1}$ points directly outward at some point of $S^n$ and directly inward at some point of $S^n$.

**Problem 18.** If $h: S^n \to S^n$ is nullhomotopic, then $h$ has a fixed point and $h$ maps some point $x$ to its antipode $-x$.

**Problem 19.** Show that the “figure-eight” $S^1 \vee S^1$ is a deformation retract of doubly-punctured plane $\mathbb{R}^2 - \{p, q\}$, $p \neq q$. 
**Problem 20.** Let \( X = S^n \vee S^n \) be the one-point union of two copies of \( S^n \). What is the fundamental group of \( X \)? (Warning: the one-point union of two simply-connected spaces is not necessarily simply connected.)

**Problem 21.** Show that \( \mathbb{R}^1 \) and \( \mathbb{R}^n \) are not homeomorphic if \( n > 1 \); and show that \( \mathbb{R}^2 \) and \( \mathbb{R}^n \) are not homeomorphic if \( n > 2 \).

**Problem 22.** Using a covering space \( X \to S^1 \vee S^1 \) of the figure-eight with finitely many sheets show that the fundamental group of \( S^1 \vee S^1 \) is not abelian.

In the following exercises, we assume that \( X \) is a union \( U \cup V \) of two connected open subsets with the property that \( U \cap V \) is connected; and \( x_0 \in U \cap V \). Denote by \( i_1: U \cap V \to U, i_2: U \cap V \to V, j_1: U \to X \) and \( j_2: V \to X \) the respective inclusions.

**Problem 23.** Suppose that the homomorphism \( i_\#: \pi_1(U \cap V, x_0) \to \pi_1(X, x_0) \), induced by the inclusion \( i: U \cap V \to X \) is trivial. Show that \( j_1 \) and \( j_2 \) induce an isomorphism:

\[
h: (\pi_1(U, x_0)/N_1) \ast (\pi_1(V, x_0)/N_2) \to \pi_1(X, x_0),
\]

where \( N_1 \) denotes the smallest normal subgroup of \( \pi_1(U, x_0) \) containing the image of \( i_1\# \) and, similarly, \( N_2 \) denotes the smallest normal subgroup of \( \pi_1(V, x_0) \) containing the image of \( i_2\# \).

**Problem 24.** Suppose that \( i_2\# \) is surjective. Show that \( j_1\# \) induces an isomorphism

\[
h: \pi_1(U, x_0)/M \to \pi_1(X, x_0)
\]

where \( M \) is the smallest normal subgroup of \( \pi_1(U, x_0) \) containing \( i_1\#(\ker i_2\#) \).

**Problem 25.** Let \( X \) be a Hausdorff space and let \( A \) be a closed path-connected subspace. Suppose that \( h: B^n \to X \) is a continuous map that sends \( S^{n-1} \) to \( A \) and maps the interior \( \text{Int}(B^n) \) bijectively onto \( X - A \). Let \( a \) be a point of \( h(S^{n-1}) \). If \( n > 2 \), what can you say about the homomorphism of \( \pi_1(A, a) \) into \( \pi_1(X, a) \), induced by the inclusion of \( A \) in \( X \)?

**Problem 26.** Let \( G \) be a finitely presented group. Show that there is a compact Hausdorff space \( X \) whose fundamental group is isomorphic to \( G \).

**Problem 27.** Let \( T = S^1 \times S^1 \) be the torus, and let \( x_0 = (1,1) \) be a base point. Prove:
a. Every automorphism of $\pi_1(T,x_0)$ is induced by a homeomorphism of $T$ with itself that sends $x_0$ to $x_0$.

b. If $E$ is a covering space of $T$, then $E$ is homeomorphic to either $\mathbb{R}^2$, or to $S^1 \times \mathbb{R}$ or to $T$ itself.

HINT: You may use the following: If $F$ is a free abelian group of rank 2 and $N$ is a nontrivial subgroup, then there is a basis $a_1, a_2$ for $F$ such that

i. Either $ma_1$ is a basis for $N$, for some $m \in \mathbb{N}$;

ii. or $\{ma_1, na_2\}$ is a basis for $N$, where $m$ and $n$ are positive integers.

Using the three types of covering spaces, exhibit explicitly the distinct covering maps corresponding to the Galois correspondence between subgroups of $\pi_1(T,x_0)$ and based equivalence classes of based covering spaces of $(T,x_0)$.

Problem 28. Let $q: X \rightarrow Y$ and $r: Y \rightarrow Z$ be maps and let $p = r \circ q$.

a. Let $q$ and $r$ be covering maps. Show that if $Z$ has a universal covering space, then $p$ is a covering map.

b. Give an example where $q$ and $r$ are covering maps but $p$ is not.

Problem 29. Let $p: X \rightarrow B$ be a covering map (not necessarily regular); let $G$ be its group of covering transformations.

a. Show that the action of $G$ on $X$ is properly discontinuous;

b. Let $\pi: X \rightarrow X/G$ be the projection map. Show that there is a covering map $k: X/G \rightarrow B$ such that $k \circ \pi = p$.

Problem 30. Give examples of:

a. A group $G$ of homeomorphisms of the torus $T$ having order 2 such that $T/G$ is homeomorphic to the torus.

b. A group $G$ of homeomorphisms of the torus $T$ having order 2 such that $T/G$ is homeomorphic to the Klein bottle.