FINAL EXAM
(DATE DUE: 12/13/2000)

INSTRUCTIONS:

1. Work on ALL problems and use separate sheets of paper for different problems.
2. You are not allowed to consult with other students and/or professors. You may consult your instructor, your textbook and your notes.
3. START WORKING ON YOUR FINAL IMMEDIATELY! Do not leave to the last minute, otherwise you will run out of time.

Problem 1. Explain why a compact, oriented manifold cannot be contractible.

Problem 2. Compute the cohomology of the product $M = S^{n_1} \times \cdots \times S^{n_k}$ of spheres of dimensions n_i, $i = 1, \ldots, k$.
(HINT: You only need to find the dimension $\dim H^r(M)$ for all r.)

Problem 3. Consider the submanifolds $j_i : C_i \hookrightarrow T^n$ of the n-dimensional torus $T^n = S^1 \times \cdots \times S^1$ defined as the i-th factor $C_i = (1, \ldots, t, \ldots, 1)$, where $t \in S^1$ is placed in the i-th coordinate. Show that the map

$$\Omega^1(T^n) \to \mathbb{R}^n$$
$$\omega \mapsto \left(\int_{C_1} j_1^*\omega, \ldots, \int_{C_n} j_n^*\omega\right)$$

induces an isomorphism $H^1(T^n) \to \mathbb{R}^n$.

Problem 4. Let $\omega \in \Omega^r(M^n)$. Suppose that

$$\int_\Sigma \omega = 0,$$

for every oriented smooth manifold $\Sigma \subset M^n$ that is diffeomorphic to S^r. Show that $d\omega = 0$.
(HINT: First reduce to the case where M is an open set in $\mathbb{R}^n$.)

Definition A. A *symplectic vector space* (V, ω) is a real vector space equipped with an alternating 2-form $\omega \in \mathrm{Alt}^2(V)$. If the form is non-degenerate, then V is called a *non-degenerate* symplectic vector space.

Problem 5. Let M be an oriented closed smooth manifold of dimension $n \equiv 2 \mod 4$.

- **a:** Show that $H^{\frac{n}{2}}(M)$ has a natural structure of a non-degenerate symplectic vector space.
- **b:** Prove that the Euler characteristic $\chi(M)$ is even.

Problem 6. Let $\mathcal{B} = \{U_1, \ldots, U_r\}$ be a finite open cover of a manifold M.

- **a:** Express the Euler characteristic $\chi(M)$ of M in terms of the Euler characteristics of intersections of the form $U_I := U_{i_1} \cap \cdots \cap U_{i_k}$, where $I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, r\}$.
- **b:** Let $f : X \to Y$ be a smooth covering space of degree d, where X and Y are compact manifolds. In other words, every point of Y has a neighborhood U such that the inverse image $f^{-1}(U)$ is written as a disjoint union $V_1 \amalg \ldots \amalg V_d$ such that the restriction $f_{|V_i} : V_i \to U$ is a diffeomorphism. Show that $\chi(X) = d \cdot \chi(Y)$.

Problem 7. Let $f : E \to M$ be a smooth map between manifolds E and M satisfying the following property: *Every point $x \in M$ has a neighborhood U such that there is a diffeomorphism $\phi : p^{-1}(U) \to U \times \mathbb{R}^n$ such that the following diagram commutes*

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\phi} & U \times \mathbb{R}^n \\ {\scriptstyle p}\big\downarrow & \swarrow{\scriptstyle \pi_1} & \\ U, & & \end{array}$$

where π_1 is the projection onto the first factor. Prove that if M has a finite good cover then $f^* : H^p(M) \to H^p(E)$ is an isomorphism for all p.

Remark Read the discussion of *Poincaré duals of an oriented closed submanifold S* of a manifold M, in pages 51 and 52 of your textbook. Be sure you understand the difference between the *closed Poincaré dual* and the *compact Poincaré dual* of a compact oriented submanifold S.

Problem 8. Solve Problem 5.16, page 52 of your textbook.

Problem 9. Let M and N be smooth, compact, oriented manifolds of same dimension n and let $f : M \to N$ be a smooth map and suppose that $f^* : H^n(N) \to H^n(M)$ is not the zero map. Prove that $f^* : H^p(N) \to H^p(M)$ is injective for all p.

Problem 10. Let X be a d-dimensional, oriented manifold, and let ω be a differential form with compact support on X. You have learned how to integrate ω on X whenever ω is an d-form. In case $\omega \in \Omega_c^k(X)$, with $k \neq d$, define $\int_X \omega = 0$. Now, let $X = M \times N$, where M and N are smooth, compact, oriented manifolds of dimensions m and n, respectively. Given forms $\omega \in \Omega^k(M)$ and $\tau \in \Omega^r(N)$, with $k + r = m + n$, prove (with details) the following form of Fubini's theorem:

$$\int_{M \times N} \pi_1^* \omega \wedge \pi_2^* \tau = \left(\int_M \omega \right) \left(\int_N \tau \right),$$

where $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ are the projection maps.