MATH 639-600, ITERATIVE TECHNIQUES

Homework #2, Due February 12, 2004
Vector and matrix norms and norm inequalities

Each problem is worth 20 points. Delay of the homework is penalized by 5 pts per day. Comments, suggestions, and even solutions of some of the problems given below one can find in Chapter 2 of Hackbusch's book, Iterative solution of large sparse systems of equations, Springer-Verlag, 1994. Solve any set of problems for 100 pts.

The norm of matrices used below are subordinated (correspond) to a vector norm. We have used the following vector norms:

$$||x||_\infty := \max_{1\le i\le n}\{|x_i|\} \quad \text{and} \quad ||x||_2^2 := \sum_{i=1}^{n} |x_i|^2 := x^H x := (x,x).$$

1. (a) Let $A, B \in \mathbb{C}^{n\times n}$ be Hermitian. Prove that $AB = BA$ if and only if there is a unitary matrix Q such that both $Q^H AQ$ and $Q^H BQ$ are diagonal;
   (b) Let $A \in \mathbb{C}^{n\times n}$ be Hermitian. Prove that A is Hermitian if and only if $x^H Ax$ is real for any $x \in \mathbb{C}^{n\times n}$.

2. For $x \in \mathbb{C}^n$, prove that $||x||_\infty \le ||x||_2, \quad ||x||_2 \le \sqrt{n}||x||_\infty$.

3. Prove that for any matrix $A \in \mathbb{C}^{n\times n}$

   $||A||_\infty = \max_i \sum_{j=1}^n |a_{ij}|, \quad ||A||_2 \le \sqrt{n}||A||_\infty, \text{ and } ||A||_\infty \le \sqrt{n}||A||_2.$

4. Let $A \in \mathbb{C}^{n\times n}$. Show that:

   (a) $||A||_2 = ||A^H||_2 = ||A^T||_2 = ||\bar{A}||_2$.

   (b) $|a_{ij}| \le ||A||_2$ for all matrix entries of A.

   (c) $|a_{ij}| \le C = \mathit{const}$ for all matrix entries of A implies that $||A||_2 \le nC$.

5. (Saad, problem 35, p. 42) Let $A \in \mathbb{R}^{n\times n}$ has entries: $a_{ij} = 0$ for $i > j$, $a_{ii} = 1$ and $a_{ij} = -1$ for $i < j$ (i.e. A is a upper triangular matrix). Find its inverse and find the condition number with respect to $||\cdot||_\infty$-norm.

6. (Saad, problem 19, p. 41) (**bonus**) Let $A \in \mathbb{C}^{n\times n}$ be such that $A^H = p(A)$. Show that A is normal. Given a diagonal matrix $D \in \mathbb{C}^{n\times n}$, show that there exists a polynomial $p(t)$ of degree less than n such that $\bar{D} = p(D)$. Use this to show that a normal matrix A satisfies $A^H = p(A)$ for certain polynomial of degree less than n.