Math 131 Week-in-Review #4 (Sections 2.6-2.8)

1. The position of an object moving in a straight line is given by \( s = f(t) = t^{-1} - 2t \), where \( s \) is measured in meters and \( t \) in seconds. Find the velocity and speed of the object after four seconds using the definition of the derivative at a number (Form 1) (Form 2).

\[
\frac{f'(4)}{4} = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h}
\]

\[
\frac{f(4+h) - f(4)}{h} = \left[ \frac{1}{4+h} - 2(4+h) \right] - \left[ \frac{1}{4} - 2(4) \right]
\]

\[
= \left[ \frac{-2(4+h)}{4+h} \right] - \left[ \frac{-32}{4} \right] = \frac{1-2(16+8h+h^2)}{4+h} + \frac{32}{4}
\]

\[
= \frac{1-32-16h-2h^2}{4(4+h)} = \frac{-124 - 64h - 8h^2 + 124 + 32h}{h(4(4+h))}
\]

\[
= \frac{-33h - 8h^2}{4(4+h)} = \frac{\frac{33}{16}}{v_{\text{veloc.}}} = \frac{33}{16} \text{ m/s}
\]

\[
\lim_{h \to 0} \frac{-33-8h}{4(4+h)} = \frac{-33}{16} \text{ m/s}
\]

\[
\text{speed}
\]
2. The following limit represents the derivative of some function $f$ at some number $a$. State such an $f$ and $a$:

$$\lim_{x \to \pi/4} \frac{\tan x - 1}{x - \pi/4}$$

Looks like $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.

hyp $f(x) = \tan x$ and $a = \pi/4$.

$\Rightarrow \lim_{x \to \pi/4} \frac{\tan x - \tan \pi/4}{x - \pi/4} = 1 \Rightarrow \begin{cases} f(x) = \tan x \\ a = \pi/4 \end{cases}$
3. A pie has just been taken out of the oven and is cooling before being eaten. Suppose that \( T(m) \) is the temperature of the pie, in degrees Fahrenheit, after it has been out of the oven \( m \) minutes.

a) Interpret \( T(5) = 250 \)

\[
\text{Five min. after being taken out of the oven, the temp. of the pie is 250 \(^\circ\text{F}\).}
\]

b) Interpret \( \frac{T(3) - T(1)}{2} = -10 \). (Avg. rt. of change between 1 and 3 mins)

\[
\text{Between one and three mins. after being taken out of the oven, the temp. of the pie is decreasing on average by 10 \(^\circ\text{F}\)/min.}
\]

c) What are the units of \( T'(m) \), and what is the sign of \( T'(m) \)?

\[
\text{T}'(m) < 0 \ (\text{neg.}) \quad \text{(temp. is decreasing)}
\]

d) Interpret \( T'(5) = -8 \).

\[
\text{Five min. after being taken out of the oven, the temp. is decreasing by 8 \(^\circ\text{F}\)/min.}
\]
4. Use the definition of the derivative to find \( f'(x) \) if \( f(x) = \sqrt{5x-1} \), and then use your formula to find the equation of the line tangent to \( f(x) \) at the point (2,3).

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}
\]

\[
\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{5(x+h)-1} - \sqrt{5x-1}}{h}
= \frac{\sqrt{5x+5h-1} - \sqrt{5x-1}}{h}
\]

\[
= \frac{(\sqrt{5x+5h-1}) - (\sqrt{5x-1})}{h(\sqrt{5x+5h-1} + \sqrt{5x-1})}
= \frac{5x+5h-1 - 5x-1}{h(\sqrt{5x+5h-1} + \sqrt{5x-1})}
= \frac{5h}{h(\sqrt{5x+5h-1} + \sqrt{5x-1})}
\]

\[
= \frac{5}{\sqrt{5x+5h-1} + \sqrt{5x-1}} \quad \Rightarrow \quad \lim_{h \to 0} \frac{5}{\sqrt{5x+5h-1} + \sqrt{5x-1}} = \frac{5}{\sqrt{5x-1} + \sqrt{5x-1}}
\]

slope of tangent at \( x=2 \) \( \Rightarrow \)
\( f'(2) = \frac{5}{2\sqrt{10}-1} = \frac{5}{10} = \frac{5}{10} = \frac{5}{10} = m \)

\[
y - 3 = \frac{5}{10} (x-2) \quad \Rightarrow \quad y = \frac{5}{10} x + \frac{8}{10}
\]
5. Create a chart showing the graphical relationships between $f$, $f'$, and $f''$.

<table>
<thead>
<tr>
<th>$f$</th>
<th>$f'$</th>
<th>$f''$</th>
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<tbody>
<tr>
<td>↑</td>
<td>⊕</td>
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</tbody>
</table>
6. Consider the graph of \( f(x) \) below.

a) Arrange the derivatives at the given points from smallest to largest. (Numeric value) (Slopes)

\[
F, C, E, B, A, D
\]

b) At what points do \( f'(x) \) and \( f''(x) \) have the same sign?

\[
f'(x) > 0 \text{ and } f''(x) > 0 \Rightarrow f \text{ is min. } + f \text{ is cc up}
\]

\[
A + D \text{ (pos.)}
\]

\[
f'(x) < 0 \text{ and } f''(x) < 0 \Rightarrow f \text{ is dec. } + f \text{ is cc down}
\]

\[
F \text{ (neg.)}
\]
7. Sketch the graph of a function $f$ that is continuous everywhere, but it is not differentiable at $x = -4$ because there is no tangent line or at $x = 3$ because the tangent line is vertical. (Diff. everywhere else)
8. Sketch the **graph of the derivative** of each of the following functions.

a) 

\[ h(x) \]

- Look for: horizontal tangents, inflection points, level shifts, etc.
- \( \text{deriv} = 0 \) \( \rightarrow \) \( \text{deriv: max/min} \) \( \rightarrow \) \( \text{deriv: near} \)

\[ h \downarrow \quad \uparrow \quad \downarrow \quad \text{linear} \]

\[ h' \ominus a \quad \ominus b \quad \ominus \quad \text{constant slope} \]

- \( h' \) less than zero \( \rightarrow \) more negative
- \( h' \) greater than zero \( \rightarrow \) more positive

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b) \( k(x) \)

\[
\begin{align*}
K \text{ (level)} & \downarrow \quad \Theta \quad \oplus \\
K' & \quad \Theta \quad \ominus \\
\text{near } 0 & \quad \text{more neg.} \quad \text{less pos.} \\
& \text{near } 0 \\
\text{v.a.} & \quad \text{(discont.)}
\end{align*}
\]
9. a) Sketch a function whose slope is positive and decreasing. (fxn is incr)

b) Sketch a function whose slope is getting less negative.

c) Sketch a function that is increasing whose slope is also increasing.
10. Sketch the graph of a function that satisfies all of the following conditions: 

- $f'(1) = f'(-1) = 0$  
  horizontal tangents at $x=1$ and $x=-1$
- $f'(x) < 0$ if $|x| < 1$  
  f decreases on $(-1, 1)$
- $f'(x) > 0$ if $1 < |x| < 2$  
  f increases on $(-2, -1)$ and $(1, 2)$
- $f'(x) = -1$ if $|x| > 2$  
  f has constant slope of $-1$ on $(-\infty, -2)$ and $(2, \infty)$
- $f''(x) < 0$ if $-2 < x < 0$  
  f is concave down on $(-2, 0)$
- inflection point at $(0, 1)$  
  Concavity changes

Shape: 

- Linear slope is $-1$
- Concave down on $(-\infty, -2)$ and $(2, \infty)$
11. Let $F$ be an antiderivative of the function $f$ whose graph is shown below.

\[ \text{graph of } f \]

a) Where is $F$ increasing?
   \[ (0, 4) \text{ and } (8, \infty) \]

b) Where is $F$ decreasing?
   \[ (-\infty, 0) \text{ and } (4, 8) \]

c) Where is $F$ concave up?
   \[ (-\infty, 2) \text{ and } (6, 9) \]

d) Where is $F$ concave down?
   \[ (2, 4) \text{ and } (9, \infty) \]
e) Use the above information to sketch a graph of $F$ if $F(0) = 0$. 

Shape

$F$

$\downarrow \uparrow \uparrow \downarrow \downarrow \uparrow \uparrow$

$U \ U \ \wedge \ \wedge \ U \ U \ \wedge

$F$

$\uparrow \downarrow \uparrow \downarrow \uparrow \downarrow \uparrow \downarrow$
12. The table below gives values of $P(t)$, the population of a small city in Texas in year $t$. (Midyear estimates are given.)

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>$P(t)$</td>
<td>29,036</td>
<td>29,672</td>
<td>32,300</td>
<td>36,205</td>
<td>38,260</td>
</tr>
</tbody>
</table>

a) Find the average rate of growth from 1996 to 2000, and interpret your answer. (Round your final answer to the nearest integer, if necessary.)


Between 1996 and 2000, the population increased on average by 1633 people/yr.

b) Estimate and interpret $P'(2000)$. (Round your final answer to the nearest integer, if necessary).

$$\text{Average of 2 avg. rates of change}$$

1998 to 2000:
$$\frac{38,260 - 32,300}{2000 - 1998} = 2525$$

2000 to 2002:
$$\frac{38,260 - 36,205}{2002 - 2000} = 1027.5$$

$$\frac{2525 + 1027.5}{2} = 1821.25 \approx 1821 \text{ people/yr.}$$

In 2000, the population increased by 1821 people/yr.