Math 144A Week-in-Review #7 (Exam 2 Review: Sections 4.1-4.5 and 5.1-5.6)

Note: This collection of questions is intended to be a brief overview of the exam material (with emphasis on sections 5.5 and 5.6). When studying, you should also rework your notes, the previous week-in-reviews for this material, as well as your suggested and online homework.

1. Use the graph below to state the absolute and local/relative maximum and minimum values of the function $f$ on the following intervals, if any exist. (Graph courtesy of Kendra Kilmer)

   a) $(-\infty, \infty)$
   - No abs. max
   - Local max is 4.5 ($@ x = 4$)
   - No abs. min
   - Local min is -1.4 ($@ x = 1$)

   b) $[-0.5, 1)$
   - Abs. max is 2 ($@ x = -0.5$)
   - No abs. min (open $@ x = 1$)
   - No local max/min (Can't occur at ends)

   c) $[3, 4.5)$
   - Abs. max is 4.5 ($@ x = 4$)
   - No abs. min (open $@ x = 4.5$)
   - Local max is 4.5 ($@ x = 4$) and no local min

   d) $(-2, 3)$
   - No abs. max
   - Local min is -1.4 ($@ x = 1$)
   - No abs. min
   - No local max

2. Find the end behavior of the function $f(x) = ax^2 + bx^4 - cx^3 - dx + 6$ if $a$, $b$, $c$, and $d$ are constants with $a > 0$, $b < 0$, $c < 0$, and $d > 0$.

   a) $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} bx^4 \to [-\infty]$ because $- (4)^4$

   b) $\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} bx^4 \to [-\infty]$ because $- (4)^4$
3. A company manufactures and sells $x$ radios per year. They can sell 700 radios each year when price is $1.50 per radio. For each decrease in price of $25, they can sell an additional 50 radios each year. Using calculus, determine how many radios should be produced to maximize revenue. What is the maximum revenue?

Need $R$: $R = x \cdot p \Rightarrow \text{need } p: (-700, 150) \cup (750, 125) \\
\Rightarrow m = -\frac{1}{2} \Rightarrow p - 150 = -\frac{1}{2}(x - 700) \\
\Rightarrow p = -\frac{1}{2}x + 500 \\
R = -\frac{1}{2}x^3 + 500x$

1. Maximize: $R(x) = -\frac{1}{2}x^3 + 500x$

2. Interval: $x \geq 0, p \geq 0 \\
\Rightarrow -\frac{1}{2}x + 500 \geq 0 \Rightarrow 500 \geq \frac{1}{2}x \Rightarrow \frac{1000}{x} \geq x \\
\Rightarrow \left[0, 1000\right]$

3. Opt. tech.: $R'(x) = -x + 500$ \\
$R' = 0 \Rightarrow (x = 500)$, (c.v. in interval) \\
$R'$ DNE? Never \\
Closed Interval Method: $R(0) = 0, R(1000) = 0$ \\
$R(500) = 125,000 \Rightarrow \text{abs. max at } x = 500$.

4. Answer: Sell 500 radios to max. rev. 
Max rev. is $125,000$

4. Find the derivative of the function $f(x) = \frac{\ln(e^{5x^4} + 7(3x^3 + 3)^8)}{\log_4 x} + 9x^6 + \pi x^3$:

$f'(x) = \left((\ln(e^{5x^4} + 7(3x^3 + 3)^8))\left(\frac{1}{e^{5x^4} + 7(3x^3 + 3)^8}\right)e^{5x^4}(20x^3) + 7(3x^3 + 3)^8(\ln 7)(8(3x^3 + 3)^7(6x))\right)$

$+ q^x(12x) + 6x^2(q^x \ln q) + 3\pi x^2$
5. Use the graph of \( f(x) \) below to find the following:

a) Where is \( f(x) < 0? \) \((\) below x axis\( \))
\[ (a, c) \text{ and } (e, \infty) \]

b) Find the critical values of \( f(x) \).
\[ f' = 0 \quad \text{and} \quad f \text{ defined} \]
\[ f' \text{ DNE} \]
\[ \{ b, 0, d \} \]

c) Where is \( f''(x) > 0? \) \((\) f cc up\( \))
\[ (-\infty, a), (a, b), \text{ and } (b, c) \]

d) Find the partition numbers of \( f'(x) \).
\[ f' = 0 \quad \text{and} \quad f \text{ defined} \]
\[ \{ a, b, 0, d \} \]

e) Where is \( f'(x) < 0? \) \((\) f decr.\( \))
\[ (-\infty, a), (b, d), \text{ and } (d, \infty) \]

f) Where is \( f'(x) \) decreasing?
\[ \text{Slope of } f \text{ decr.? (i.e. } f \text{ cc down?)} \]
\[ (c, \infty) \]

g) Find and classify any local/relative extrema of \( f(x) \).
\[ \text{look at critical values} \]
\[ \text{local max at } x = b \]
\[ \text{local min at } x = 0 \]
\[ \text{local max at } x = d \]
6. A rectangular storage container with an open top is to have a volume of 10 m$^3$. The length of its base is twice the width. Material for the base costs $8 per square meter. Material for the sides costs $4 per square meter. Use calculus to find the cost of materials for the cheapest such container.

\[
\text{Min cost} = 8(2w^3) + 4(2wh)(2) + 4(hw)(2)
\]

\[
\text{Known volume: } 2w^2h = 10 \Rightarrow h = \frac{10}{2w^2} = \frac{5}{w^2}
\]

\[
\begin{align*}
\text{Minimize: } C(w) &= 8(2w^3) + 16w \left( \frac{5}{w^2} \right) + 8w \left( \frac{5}{w^2} \right) \\
&= 16w^3 + \frac{80}{w} + \frac{40}{w} = 16w^3 + \frac{120}{w} = C(w)
\end{align*}
\]

\[
\begin{align*}
\text{Interval: } & w > 0, \quad h > 0 \\
& 4 \frac{5}{w^2} > 0 \Rightarrow w > 0 \text{ (width)} \Rightarrow (0, \infty)
\end{align*}
\]

\[
\begin{align*}
\text{Opt. tech: } C'(w) &= 32w - \frac{120w^{-2}}{} = 32w - \frac{120}{w^2} = 32w^3 - 120 \\
C' &= 0 \Rightarrow 32w^3 - 120 = 0 \Rightarrow 32w^3 = 120 \Rightarrow w^3 = \frac{120}{32} \\
&= \left( \frac{32}{120} \right)^{1/3} \\
&= \frac{\sqrt[3]{15^4}}{4} \approx 1.6
\end{align*}
\]

* One c. v. and open interval $\Rightarrow$ 1st Deriv. Test.

\[
\begin{align*}
C &= 0 \quad \uparrow \\
&= 3.18 \quad \uparrow \\
&\approx 1.6
\end{align*}
\]

\[
\text{Abs. min. at } w = \frac{3.18}{3} \approx \frac{1.6}{3}
\]

\[
\text{Answer: (plug into C)} \\
C = 16w^3 + \frac{120}{w} = 16\left( \frac{3}{15^4} \right)^3 + \frac{120}{\frac{3}{15^4}} = 115.86
\]

7. If $y = t^2 + 2t$ and $t = p + \ln p$, find $dy/dp$.

\[
\frac{dy}{dp} = \frac{dy}{dt} \cdot \frac{dt}{dp}
\]

\[
\Rightarrow (2t + 2)(1 + \frac{1}{p}) \quad \star \text{Substitute for } t \Rightarrow
\]

\[
\frac{dy}{dp} = (2(p + \ln p) + 2)(1 + \frac{1}{p})
\]
8. A company has a price-demand function of $p = \frac{-x^2}{2} + 96$ where $x$ is the number of units sold at a price of $p$ dollars per unit. The current price per unit is $35$.

a) Find the percentage change in demand if price increases by $5$.

\[ \text{E}(p) = -p \frac{f'(p)}{f(p)}. \]

\[ \Rightarrow \text{need } x = f(p) \text{ first } \Rightarrow 2p = -x^2 + 192 \]

\[ \Rightarrow x^2 = 192 - 2p \Rightarrow x = \sqrt{192 - 2p} \text{ (pos. root) demand} \]

\[ E(p) = -p \left( \frac{1}{2} (192 - 2p)^{-\frac{1}{2}} \right) \left( \frac{-1}{2} \right) = \frac{p (192 - 2p)^{-\frac{1}{2}}}{(192 - 2p)^{\frac{1}{2}}} = \frac{p}{(192 - 2p)^{\frac{1}{2}}} \]

\[ E(35) = 0.2869 \]

\[ \% \text{ chg price: } \frac{40 - 35}{35} = 0.1429 = 14.29\% \]

\[ \% \text{ chg dem: } (0.2869)(14.29\%) \]

\[ \Rightarrow \text{dem. is by 4.10%} \]

b) At the current price, how will revenue change if the price does increase by $5$?

\[ E(35) = 0.2869 < 1 \text{ so dem. is inelastic} \]

\[ \Rightarrow \text{Rev. will increase} \]

c) What should the company charge per unit in order to maximize its revenue?

\[ \Rightarrow E(p) = 1. \]

\[ \Rightarrow \frac{p}{192 - 2p} = 1 \Rightarrow p = 192 - 2p \Rightarrow 3p = 192 \]

\[ \Rightarrow p = \$64 \]
9. Assuming the graph shown below is $f'(x)$ and $f(x)$ is continuous on its domain of $(-\infty, d) \cup (d, \infty)$, find the following:

a) Find the critical values of $f(x)$.

$f' = 0$ and $f$ Defined

$a, e, g$

b) Find the partition numbers of $f'(x)$.

$f' = 0$

$a, d, e, g$

f' DNE

c) Where is $f(x)$ increasing? ($f' > 0$)

$(-\infty, a)$ and $(e, g)$

d) Find and classify any local/relative extrema of $f(x)$.

$f' = 0 \Rightarrow f$ DNE, $f$ defined, $f'$ changes sign

i.e. look at cut values $a, e, g$ and see if $f'$ changes sign

$f' + = - + +$

$f' \uparrow \downarrow e \uparrow \downarrow g \downarrow$

local max at $x = a$ and $x = g$

local min at $x = e$

e) Where is $f(x)$ concave up?

$f'$ increasing

$(b, d)$ and $(0, f)$

f) Where is $f''(x) < 0$?

$f'$ decreasing

$(d, 0)$ and $(f, \infty)$

g) Find the partition numbers of $f''(x)$.

$f'' = 0$

$a, c, d, 0, f$

f' DNE

h) Where does $f(x)$ have inflection points (if any)?

$i) \text{ Find the critical values of } f''(x)$.

Note: $cv \text{ off } f' \Rightarrow cv \text{ off } f''$

$f' = 0$

f' DNE

f defined

$f'' = 0$

f'' DNE

f' defined

$\Rightarrow c, 0, f$
10. Ryan and Joe produce personalized watches. Suppose their cost function is given by \( C(x) \), where \( x \) is the number of watches made and \( C(x) \) is the total cost of producing \( x \) watches in dollars. Use the following information to help you answer the questions below.

| \( C(21) = \$378 \) | \( C'(21) = \$168 \) |
| \( C(22) = \$506 \) | \( C'(22) = \$88 \) |
| \( C(23) = \$554 \) | \( C'(23) = \$8 \) |
| \( C(24) = \$522 \) | \( C'(24) = -\$72 \) |

a) Find the exact cost of the 23\textsuperscript{rd} watch.

\[ C(23) - C(22) = \$48 \]

b) Approximate/estimate the cost of the 23\textsuperscript{rd} watch.

\[ C'(22) = \$88 \]

c) Find the rate of change of cost when 22 watches are produced.

\[ C'(22) = \$88/\text{watch} \]

d) Find the exact cost if 24 watches are produced.

\[ C(24) = \$522 \]

e) Approximate/estimate the cost if 24 watches are produced.

\[ C(23) + C'(23) = \$562 \]

f) Find the average cost per watch if 23 watches are produced.

\[ \frac{C(23)}{23} = \$24.09 \]

g) Find the marginal cost if 21 watches are produced, and interpret your answer.

\[ C'(21) = \$168/\text{watch} \]

When 21 watches are produced, cost is increasing by \( $168/\text{watch} \).
11. a) If we know \( f(3) = 4 \) and \( f'(3) = 0 \), and we also know that \( f'' \) is continuous everywhere and \( f''(3) = -6 \), then what (if anything) can we conclude about the behavior of \( f \) at \( x = 3 \)?

\( x = 3 \) is crit. value of \( f \) because \( f(3) \) exists & \( f'(3) = 0 \)

\( \Rightarrow \) Attempt to use 2nd Deriv. Test

Since \( f''(3) = -6 \rightarrow f \) is \( \bigcap \rightarrow \) local/relative max at \( x = 3 \)

b) If we know \( f(5) = 1 \) and \( f'(5) = -2 \), and we also know that \( f'' \) is continuous everywhere and \( f''(5) = 8 \), then what (if anything) can we conclude about the behavior of \( f \) at \( x = 5 \)?

\( x = 5 \) is not a crit. value of \( f \) since \( f'(5) = -2 \)

\( \Rightarrow \) no local/relative extrema at \( x = 5 \).

12. Find the equation of the line tangent to the function \( f(x) = 6 \ln x^4 + 3x - e^{-x} \) at \( x = 1 \).

Slope & point

\( m = f'(1) \)

\[
f'(x) = 6 \left( \frac{1}{x^4} \right) (4x^3) + 3 - e^{-x}(1) = \frac{24}{x} + 3 - e^{-1} \]

\( f'(1) = 24 + 3 - e^0 = 27 - 1 = 26 \Rightarrow m = 26 \)

\( x = 1 \Rightarrow y = f(1) = 6 \ln 1^4 + 3(1) - e^{-1} = (6 \cdot 0) + 3 - 1 = 2 \Rightarrow y = 2 \)

\( (1, 2) \Rightarrow m = 26 \)

\( y - a = m(x - 1) \Rightarrow y = 26x - 24 \)
13. Ben needs to enclose three regions as shown below using 2500 feet of fencing. Use calculus to find the dimensions of each region so that Ben obtains the largest total area possible.

Max area = \[xy + 2xy + xy = \frac{4}{3}xy\]

\[\text{Perimeter: } 4y + 8x = 2500 \Rightarrow 4y = 2500 - 8x\]
\[y = \frac{2500 - 8x}{4}\]

\[\Rightarrow 625 - 2x > 0 \Rightarrow 625 > 2x \Rightarrow x < 312.5\]

\[(0, 312.5)\]

3. Opt. Tech: \[A'(x) = 2500 - 16x\]
\[A' = 0 \Rightarrow 2500 - 16x = 0 \Rightarrow x = 156.25\]

4. Answer:
\[x = 156.25 \Rightarrow y = 625 - 2(156.25)\]
\[= 312.5\]

\[\Rightarrow 2 \text{ regions: } 156.25 \times 312.5\]
\[1 \text{ region: } 312.5 \times 312.5\]

14. Find the derivative of \[f(x) = (8e^{3x^2 + 4x})^{3} \left(\frac{\log(x+5)}{\text{messy}}\right)^{2/3}\]

\[f'(x) = \left(8e^{3x^2 + 4x} + 14x\right) \left[\frac{2}{3} \left(\log(x+5)^{2/3} \left(\frac{1}{\ln10} \left(\frac{1}{6x+5}(6)\right)\right)\right)\right] + \]
\[3 \sqrt{(\log(x+5)^{2}} \left[8e^{3x^2 + 4x} \left(6x+4\right) + 14x(\ln14)\right]\]
15. Find the horizontal asymptotes of the following functions if any exist:

a) \( f(x) = \frac{ax^3 + bx - c}{4x^2 + 2} \), where \( a, b, \) and \( c \) are constants with \( a < 0, b > 0, \) and \( c > 0. \)

\[
\lim_{x \to \infty} \frac{ax^3 + bx - c}{4x^2 + 2} = \lim_{x \to \infty} \frac{ax^3 + bx}{4x^2 + 2} = \lim_{x \to \infty} \frac{ax^3}{4} = \frac{-\infty}{4} \rightarrow \frac{3}{4}c 
\]

\Rightarrow \text{No H.A.}

b) \( f(x) = \frac{2e^{-4x} - e^{5x}}{e^{2x} - 6e^x + 8} \)

\[
\lim_{x \to \infty} \frac{2e^{-4x} - e^{5x}}{e^{2x} - 6e^x + 8} = \lim_{x \to \infty} \frac{2e^{-4x}}{e^{2x}} - \frac{e^{5x}}{e^{2x}} = \frac{0 - 0}{0 + 8} \rightarrow \infty
\]

\Rightarrow \text{No H.A.}

c) \( f(x) = \frac{e^x - 7e^{-3x}}{6e^x + 2e^{-x} - 4e^{-3x}} \)

\[
\lim_{x \to \infty} \frac{e^x - 7e^{-3x}}{6e^x + 2e^{-x} - 4e^{-3x}} = \lim_{x \to \infty} \frac{e^x}{6e^x} - \frac{7e^{-3x}}{6e^x} = \frac{0 - 0}{0 + 4} \rightarrow 0
\]

\Rightarrow y = 0 \text{ and } y = \frac{7}{4}
16. Use calculus, if possible, to find the absolute maximum and minimum values of the function \( f(x) = \frac{x-2}{(x-4)^2} \) on each of the following intervals, if they exist.

a) \([-2,3]\)

\( f'(x) = \frac{(x-4)^2(1) - (x-2)(2(x-4))(1)}{(x-4)^2} \)
\( f'(x) = \frac{(x-4)^2 - 2(x-2)(x-4)}{(x-4)^2} \)
\( f'(x) = \frac{(x-4)(x-4 - 2(x-2))}{(x-4)^4} \)
\( f'(x) = \frac{x-4 - 2x + 4}{(x-4)^3} \)
\( f'(x) = \frac{-x}{(x-4)^3} \)

\( f(-2) = -\frac{1}{4} \)
\( f(0) = -\frac{1}{8} \)
\( f(3) = 1 \)

\( \Rightarrow \) Abs. max is 1
Abs. min is \(-\frac{1}{8}\)

b) \([1,3]\)

\( f(1) = -\frac{1}{4} \)
\( f(3) = 1 \)

\( \Rightarrow \) Abs. max is 1
Abs. min is \(-\frac{1}{4}\)

c) \([-5,5]\)

* not continuous on \([-5,5]\) ⇒ sketch
* Look at crit. value \( x=0 \) also.
* Be careful with zoom on calc.

\( \text{No abs. max} \)
Abs. min is \(-1.25\)
(at \( x=0 \))

\( \text{NOTE: Check with your instructor about this case.} \)

\( \text{d) } (-2,3) \)

* not closed ⇒ sketch
* Look at crit. value \( x=0 \) also.

\( \text{No abs. max} \)
Abs. min is \(-1.25\)
(at \( x=0 \))

\( \text{NOTE: Check with your instructor about this case.} \)
17. Sketch the graph of a function that satisfies all of the given conditions.
   - Domain of $f$: $(-\infty, 0) \cup (0, \infty)$
   - $f(-5) = 8$ and $f(2) = -2$
   - $f'(4) = 0$
   - $f'(x) > 0$ on $(-\infty, -5), (2, 4)$
   - $f'(x) < 0$ on $(-5, 0), (0, 2), (4, \infty)$
   - $f''(x) > 0$ on $(-\infty, -2), (0, 3)$
   - $f''(x) < 0$ on $(-2, 0), (3, \infty)$
   - vertical asymptote $x = 0$
   - $\lim_{x \to \pm \infty} f(x) = 1$ (H. A.)

18. Find any holes and vertical asymptotes of the curve $y = \frac{(x-2)(x+3)(x-7)^2}{(x-7)^3(x-4)(x+3)}$ if they exist. If there are vertical asymptotes, use limits to describe the behavior near each vertical asymptote.
   - $\lim_{x \to 7^-} f(x) = -\infty$
   - $\lim_{x \to 7^+} f(x) = \infty$
   - $\lim_{x \to 4^-} f(x) = \infty$
   - $\lim_{x \to 4^+} f(x) = -\infty$
   - $\lim_{x \to -3} f(x) = \text{hole at } x = -3$
   - V.A. at $x = 7$ and $x = 4$
19. Assuming the graph shown is \( f''(x) \), and \( f(x) \) is continuous on its domain of \((a, c) \cup (c, \infty)\) and \( f'(x) \) is continuous on its domain of \((a, c) \cup (c, \infty)\), find the following:

a) Find the partition numbers of \( f''(x) \).

\[ f'' = 0 \quad f'' \text{ DNE} \]
\[ \{a, c, d, f\} \]

b) Where is \( f'(x) \) increasing? \( (f'' > 0) \)

\( (d, f) \)

(c) Where is \( f(x) \) concave down? \( (f'' < 0) \)

\( (a, c), (c, d), \) and \( (f, \infty) \)

d) Where does \( f(x) \) have inflection points (if any)?

\[ f'' = 0 \text{ on } f'' \text{ DNE, } \forall f \text{ defined, } f'' \text{ changes sign} \]

i.e. look at \( d, f \) and see if \( f'' \) changes sign.

\[ x = d \text{ and } x = f \]

e) Find \( f'''(e) \).

Horizontal tangent at \( x = e \Rightarrow f'''(e) = 0 \).

f) Find and classify any local/relative extrema of \( f'(x) \).

Note: \( f \) has local extrema when \( f' = 0 \) or \( f' \text{ DNE, } f \text{ defined, } f' \text{ changes sign} \)

so \( f' \) has local extrema when \( f'' = 0 \) or \( f'' \text{ DNE, } f' \text{ defined, } f'' \text{ changes sign} \)

\[ f'' = - + \quad + - \]

\[ f' \uparrow d \uparrow \quad \uparrow F \downarrow \Rightarrow \text{local min at } x = d \]

local max at \( x = f \).
20. Determine where the function $f(x) = \frac{\ln x}{x}$ is increasing/decreasing and concave upward/downward, as well as where any relative extrema or inflection points occur.

A. $f'(x)$:

0. Domain: $(0, \infty)$ bloc of $\ln x$

1. Partition #s of $f'$: $f'(x) = \frac{x (\ln x) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2}$

2. $f' = 0 \Rightarrow 1 - \ln x = 0 \Rightarrow 1 = \ln x \Rightarrow e^1 = e^{\ln x} \Rightarrow e = x$

3. $f'$ DNE? $\Rightarrow x = 0$ (Really $x = 0$ bloc of $\ln x$)

4. Critical values of $f$:

   $\Box x = e$

   Max. at $(0, e)$
   
   Den. on $(e, \infty)$
   
   Local max at $x = e$ is 0.3679
   
   $\uparrow f(e)$

B. $f''(x)$:

0. Domain: $(0, \infty)$

1. Partition #s of $f''$: $f''(x) = \frac{x^2(-\frac{1}{x}) - (1 - \ln x)(2x)}{(x^2)^2} = \frac{-x - 2x + 2x \ln x}{x^4}$

   $= \frac{-3x + 2x \ln x}{x^4}$

   $\Rightarrow x = e^{\frac{3}{2}}$

2. $f''$ DNE? $\Rightarrow x = 0$ (Really $x = 0$ bloc of $\ln x$)

3. Sign chart of $f''$:

   $\downarrow f^\prime$:

   $\Box x = e^{\frac{3}{2}}$

   $\Rightarrow f''$ is CC down on $(0, e^{\frac{3}{2}})$

   CC up on $(e^{\frac{3}{2}}, \infty)$

   Inf. pt. at $x = e^{\frac{3}{2}}$ is $(e^{\frac{3}{2}}, 0.3347)$