Math 142 Week-in-Review #5 (Sections 4.4, 4.5, and 5.1)

1. Find the derivative of each of the following functions. Do not simplify your answers.

a) \( f(x) = \sqrt{\ln(7x^2 - 5)} \left[ \log(x^3 + 9^x) \right] \)

\[
f'(x) = \left[ \frac{1}{2} \ln(7x^2 - 5) \right] \left[ \frac{1}{x^3 + 9^x} \right] \left[ \frac{3x^2 + 9^x \ln 9}{\ln 10} \right] \left[ \frac{3}{7x^2 - 5} \right] \left[ \ln(7x^2 - 5) \right]
\]

b) \( f(x) = \frac{\log_8(6x^2 + \pi)}{7(e^x - 3)} \)

\[
f'(x) = \left[ \frac{1}{(6x^2 + \pi)} \right] \left[ \ln 8 \right] \left[ \frac{1}{\ln e^x} \right] \left[ 12x \right] - \frac{\left( \log_8(6x^2 + \pi) \right) \left[ \frac{7(e^x - 3)}{7x(e^x)} \right]}{\left( e^x - 3 \right)^2}
\]
c) \( f(x) = \sqrt{(e^{3x^2 \ln(x)})^2} \) \\
Chain, product \\

\[
f' = \frac{2}{3} \left( e^{3x^2 \ln(x)} \right)^{\frac{3}{2}} \left[ e^{3x^2 \ln(x)} \left( \frac{1}{x} \right) + (\ln(x))(e^{3x^2 \ln(x)}) \right]
\]

2. List the three conditions necessary for a function \( f(x) \) to have a relative maximum or minimum at \( x = c \).

1. \( f' = 0 \) or \( f' \) does not exist
2. \( f \) is defined
3. \( f' \) changes sign
3. A local coffee shop has a weekly price demand equation of \( p = 30 - \frac{x}{20} \), where \( p \) is the price (in dollars) for a cup of coffee and \( x \) is the number of cups sold.

a) If the current price for a cup of coffee is $5.00, describe the effect on demand if price increases by 12%. Is demand elastic, inelastic, or unit elastic at the current price level?

\[
\frac{\partial p}{\partial x} = \frac{1}{20} \Rightarrow \frac{20p}{600-20p} \equiv \frac{600-20p}{f(p)} \]

\[
\% \text{ change in demand} = E(p) \times \% \text{ change in price} = \frac{-p f'(p)}{f(p)} = -\frac{p}{600-20p} \Rightarrow E(5) = 0.2 < 1 \quad \text{inelastic}
\]

\[
\% \text{ change in demand} = (0.2)(12\%) = 2.4 \% \Rightarrow \text{Demand down by 2.4\%}
\]

b) At what price level is the shop’s revenue maximized?

\[
E(p) = 1 \Rightarrow \frac{20p}{600-20p} = 1 \Rightarrow 20p = 600 - 20p
\]

\[
\Rightarrow \quad 40p = 600 \quad \Rightarrow \quad p = \$15
\]

(c) On what interval of \( p \) is demand elastic?

\[E(p) > 1 \]

\[
E(p) = \frac{20p}{600-20p} > 1 \Rightarrow 20p > 600 - 20p
\]

\[
40p > 600 \quad \Rightarrow \quad p > 15
\]

* Upper bound for price when elastic!*

\( x \geq 0 \) (substituting \( x \))

\[
600 - 20p \geq 0 \quad \Rightarrow \quad 600 \geq 20p \Rightarrow 30 \geq p
\]

\[
(15, 30)
\]
4. For each of the following functions, find any critical values, the intervals where the function is increasing/decreasing, and any relative extrema. Be sure to classify the type of each extrema.

a) \( f(x) = x - \ln x \)

1. **Domain:** \((0, \infty)\)  
   - bc \( x > 0 \) for \( \ln x \)

2. **Partition #’s of \( f’ \):**
   - \( f’ = 1 - \frac{1}{x} \)
   - Critical values of \( f \): \( x = 1 \)

3. **Sign chart of \( f’ \):**
   - Decreasing on \((0, 1)\)
   - Decreasing on \((1, \infty)\)
   - Relative (local) min \( @ x = 1 \)

b) \( f(x) = x^2 e^{-x} \)

1. **Domain:** \((-\infty, \infty)\)

2. **Partition #’s of \( f’ \):**
   - \( f’ = x^2 [e^{-x}(-1)] + e^{-x}[2x] \)
   - Critical values of \( f \): \( x = 0 \) \( x = 2 \)

3. **Sign chart of \( f’ \):**
   - Decreasing on \((-\infty, 0)\) and \((2, \infty)\)
   - Max. on \((0, 2)\)
   - Relative (local) min \( @ x = 0 \)
   - Relative (local) max \( @ x = 2 \)
c) \( f(x) = \frac{3x^2 - 2x}{(x - 4)^2} \)

\[ \text{mess} \]

1. **Domain:** \((-\infty, 4) \cup (4, \infty)\)

2. **Partition & \(f'\):**
   \[ f' = \frac{(x - 4)^2 \left[ (6x - 2) - (3x^2 - 2x) \right]}{(x - 4)^2} \frac{(x - 4)(1)}{(x - 4)^2} \]
   \[ \Rightarrow \frac{(x - 4) \left[ (x - 4)(6x - 2) - (6x^2 - 4x) \right]}{(x - 4)^3} \]
   \[ = \frac{(x^2 - 24x + 8 - 6x^2 + 4x)}{(x - 4)^3} = -22x + 8 = f' \]

3. **Critical Values of \(f\):**
   \[ x = \frac{8}{22} \quad \text{(in domain)} \]

4. **Sign Chart of \(f'\):**

\[ f' \theta \bullet \oplus \odot \]

\[ f \downarrow \frac{8}{22} \uparrow 4 \downarrow \]

**Decr. on** \((-\infty, \frac{8}{22})\) **and** \((4, \infty)\)

**Inc. on** \((\frac{8}{22}, 4)\)

Rel. (local) **min** at \(x = \frac{8}{22}\)

\( \text{Is } f \left( \frac{8}{22} \right) \approx -0.25 \)
5. Suppose a company that makes deluxe toasters has a weekly demand equation given by $p(x) = 150e^{-0.02x}$, where $p$ is the price in dollars when $x$ toasters are sold. Determine where the company's revenue is increasing.

$$R = x \cdot p = 150xe^{-0.02x}$$

1. **Domain:** $[0, \infty)$  
   *because $x \geq 0$ for # of toasters*

2. **Partition #'s of $R$:**
   
   $$R'(x) = 150xe^{-0.02x} \left[ e^{-0.02x} (-0.02) \right] + e^{-0.02x} [150]$$

   $$= 150e^{-0.02x} (-0.02x + 1)$$

   $R' = 0 \Rightarrow -0.02x + 1 = 0 \Rightarrow x = 50$

3. **Critical values of $R$:** $x = 50$

4. **Sign chart of $R$:**

   $$R' \begin{array}{c|c|c}
   & < 0 & > 0 \\
   \text{num. m (0, 50)} & \downarrow & \uparrow \\
   R & 0 & 50 \\
   \end{array}$$

   **always positive.**
6. Briefly state the difference between partition numbers and critical values of a function $f$.

Partition #’s of $f'$ are where $f'=0$ or $f'$ does not exist. Critical values are where $f'=0$ or $f'$ does not exist and $f$ is defined.

Critical values are a subset of partition #’s.

Critical values are partition #’s that are in the domain.

7. Doc and Marty’s Skateboard Shop has determined the price-demand equation for its “futuristic” skateboards to be $x + 0.10p = 37.5$, where $x$ is the number of skateboards that can be sold at a unit price of $p$ dollars. The current price for a skateboard is $150. In order to increase their revenue, what should the Doc and Marty do?

$$ E(p) = \frac{-p (-.10)}{37.5 - .10p} \Rightarrow E(150) = \frac{3}{5} < 1 \Rightarrow \text{inelastic} $$

To increase revenue, need to increase price (bc demand is inelastic.)
8. Consider a function $f$ that is smooth and continuous on its domain of $(-\infty, 2) \cup (2, \infty)$.

Also, $f'(x) = \frac{8(x+4)}{(x-2)^4}$. Find any critical values of $f(x)$, the intervals where $f(x)$ is increasing/decreasing, as well as where any relative extrema occur (be sure to specify the type of any relative extrema).

1. **Domain:** $(-\infty, 2) \cup (2, \infty)$

2. **Partition #’s of $f$:**
   - $f' = \frac{8(x+4)}{(x-2)^4}$
   - $f' = 0 \implies x = -4$
   - $f' \text{ does not exist} \implies x = 2$

3. **Critical values of $f$:**
   - $x = -4$

4. **Sign chart of $f$:**
   - $f'(-4) = 0$
   - $f' > 0$
   - $f' < 0$
   - $f \downarrow -4 \uparrow 2 \uparrow$

   **Graph:**
   - **Min. on** $(-4, 2)$ and $(2, \infty)$
   - **Dec. on** $(-\infty, -4)$
   - **Rel. (local) min** at $x = -4$
9. The elasticity of demand for a certain commodity is \( E(p) = \frac{2p^2}{1200 - p^2} \), where \( p \) is the unit price in dollars.

The current price per item is $30.

a) If this price is increased by $1.20, describe the effect on demand.

\[
\text{chg dem} = E(p) \times \% \text{chg price}
\]

\[
E(30) = \frac{2(30)^2}{1200 - (30)^2} = \frac{1800}{900} = 2
\]

\[
\% \text{chg price} = \frac{1.20}{30} \times 100 = 4\%
\]

\[
\text{chg dem} = (2)(4\%) = 24\%
\]

Dem \downarrow \text{by } 24\%.

b) For what values of \( p \) will a percentage change in price cause a smaller percentage change in demand?

\[
E(p) < 1 \quad \text{"inelastic"}
\]

\[
E(p) = \frac{2p^2}{1200 - p^2} < 1 \quad \Rightarrow \quad 2p^2 < 1200 - p^2
\]

\[
3p^2 < 1200
\]

\[
p^2 < 400 \quad \Rightarrow \quad |p| < 20
\]

\(-20 < p < 20
\]

\[
\Rightarrow (-20, 20)
\]

c) Given the current price is $30, what should the company do to increase revenue?

Since \( E(30) = 2 \), demand is elastic.

So \textbf{(decrease price)}. 


10. Sketch an example of a function that has a critical value at $x = 3$ such that $f'(3)$ does not exist and there is a relative maximum at $x = 3$.

11. If a function has a critical value at $x = 5$, does that mean it has a relative max or min at $x = 5$? Explain graphically.

\[ \boxed{\text{No}} \]

Possible to have critical value and no rel. (local) max/min.

\[ \begin{align*}
\text{C.V. when} & \quad f(5) = 0 \\
\text{f(5) defined} & \Rightarrow
\end{align*} \]

\[ \begin{align*}
\text{C.V. when} & \quad f'(5) \text{ one} \\
\text{f(5) defined} & \Rightarrow
\end{align*} \]