Math 617 - - Homework #4

Instructor - Al Boggess

Fall 1999

- (Do these but Do Not Hand In): Chapter 3 # 10, 17, 20 parts a, c, d.
- (Do this one, but Not Hand In): This problem constructs "cut-off functions". The problem is this: given a compact set K contained in an open set $\Omega \subset R^N$, construct a smooth function ϕ whose support is contained in Ω and which is 1 on a neighborhood of K. Follow this outline.

  1. For $x \in R^N$, let
  $$\psi(x) = \begin{cases} e^{\frac{1}{|x|^2-1}} & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 1 \end{cases}$$
  Show that ψ is a C^∞ function.
  2. For $\epsilon > 0$, let
  $$\psi_\epsilon(x) = \frac{1}{C\epsilon^N}\psi(x)$$
  where $C = \int_{R^N} \psi(x)\,dx$. Show that $\int_{R^N} \psi_\epsilon(x)\,dx = 1$ and that the support of ψ_ϵ is contained in the ball centered at 0 of radius ϵ.
  3. For a compact set $K \in \Omega$, let K_r be the set of all points in R^N whose distance to K is less than or equal to r. Show that r can be chosen small enough so that $K_r \subset \Omega$.
  4. Let χ_{K_r} be the characteristic function on K_r (1 on K_r, zero off K_r). Show that the function
  $$\begin{aligned} \phi(x) &= (\psi_\epsilon * \chi_{K_r})(x) \\ &= \int_{y \in K_r} \psi_\epsilon(x-y)\,dy \end{aligned}$$
  is C^∞. Also show that if ϵ is chosen small enough, then the support of ϕ is contained in Ω and that $\phi = 1$ on a neighborhood of K.

- (Hand-in Problem). Mimic the proof of the Cauchy Integral Formula to prove the following formula for all C^1 functions f: Suppose D is a bounded open set in C with oriented boundary γ, then for $z \in D$

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int\int_D \frac{\frac{\partial f(\zeta)}{\partial \bar{\zeta}}}{\zeta - z}\, d\bar{\zeta} \wedge d\zeta \\
&= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{\zeta - z} - \frac{1}{\pi}\int\int_D \frac{\frac{\partial f(\zeta)}{\partial \bar{\zeta}}}{\zeta - z}\, dx \wedge dy
\end{aligned}
$$

(where $\zeta = x + iy$). Note the special case when f is a C^1 function with compact support and D is a large disc which contains the support of f; then the above equation reads

$$f(z) = -\frac{1}{2\pi i}\int\int_D \frac{\frac{\partial f(\zeta)}{\partial \bar{\zeta}}}{\zeta - z}\, d\bar{\zeta} \wedge d\zeta$$

for $z \in C$. This formula will be important in Math 618, when we solve the inhomogeneous Cauchy-Riemann equations.

- (Hand-in Problem).

  1. Prove the following summation by parts formula: suppose $a_n \in C$ and $b_n \in C$; let $A_N = \sum_{j=0}^{N} a_j$; then for any $0 \leq p < q < \infty$

     $$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p.$$

     Hint: Rewrite the sum on the right involving $A_n b_{n+1}$ in terms of $A_{n-1} b_n$ (using a change of summation index). Why is this equation called summation by parts?
  2. Prove the following fact: suppose the partial sums A_n form a bounded sequence and suppose b_n is a sequence of real numbers with $0 \leq b_{n+1} \leq b_n$ for all $n \geq 0$ and $b_n \mapsto 0$ as $n \mapsto \infty$, then $\sum_{n=0}^{\infty} a_n b_n$ converges. *Hint:* use the summation by parts to show that the partial sums of $\sum_n a_n b_n$ form a Cauchy sequence.
  3. Use the previous result to show that $\sum_{n=1}^{\infty} z^n/n$ converges for all $|z| = 1$ *except* for $z = 1$.

- Also hand in Chapter 3 # 20, part b.