Self-Similar Groups and Groups Generated by Automata

Andrew Penland
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September 2, 2014
Basic Definitions - Words and Trees

- $X$ - finite set (e.g. $\{0, 1\}$)

- $X^n =$ words of length $n$ in $X$

- $X^{[d]} = \bigcup_{i=0}^{d} X^i =$ words of length $d$ or less

- $X^{(d)} = \bigcup_{i=0}^{d-1} X^i =$ words of length less than $d$
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Words and Trees

$X^{(d)} \iff |X|$-ary tree with $d$ levels

$X^{[d]} \iff |X|$-ary tree with $d + 1$ levels

Figure: $X^{[3]} = X^{(4)}$
\[ X^* = \bigcup_{i=0}^{\infty} X^i = \text{all words in } X \]

\[ \cong \text{infinite } |X|\text{-regular tree rooted at } \epsilon \]

Vertices - elements of \( X^* \)

Edges - \( \{w, wx\} \) (\( w \in X^*, x \in X \))
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Automorphisms of $X^*$

**Definition**

An *automorphism* of $X^*$ is a graph automorphism of the rooted tree $X^*$.

$\text{Aut}(X^*)$ - group of all automorphisms of $X^*$

*Groups generated by automata* – special subgroups of $\text{Aut}(X^*)$
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*Groups generated by automata* – special subgroups of $\text{Aut}(X^*)$
A non-initial Mealy automaton $A$ is a 4-tuple $A = (S, X, P, T)$ where

- $S$ - set of states
- $X$ - alphabet
- $P : S \times X \rightarrow X$ - print function
- $T : S \times X \rightarrow S$ - transition function
Indicates $P(s_1, x) = y$ and $T(s_1, x) = s_2$. 
Automaton - used as “machine” which transforms elements of $X^*$

For each $s \in S$, there is an induced function $T_s : X \to X$ given by $T_s(x) = T(s, x)$

Choosing an initial state $s_0 \in S$, $T_{s_0}$ extends to a function $X^* \to X^*$. 
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States Give Functions – Example
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\[
\begin{align*}
T_a(00) &= 10 \\
T_a(10) &= 01 \\
T_a(01) &= 11 \\
T_a(11) &= 00
\end{align*}
\]

\[
\begin{align*}
T_e(00) &= \\
T_e(10) &= \\
T_e(01) &= \\
T_e(11) &=
\end{align*}
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\[ \{0, 1\}^2 \iff \mathbb{Z}/4\mathbb{Z} \]

\begin{align*}
00 & \iff 0 \\
10 & \iff 1 \\
01 & \iff 2 \\
11 & \iff 3
\end{align*}
$T_a : \mathbb{Z}/4\mathbb{Z} \rightarrow \mathbb{Z}/4\mathbb{Z}$

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\{0, 1\}^2 \iff \mathbb{Z}/4\mathbb{Z}
\]

\[
T_a(0) = 1 \\
T_a(1) = 2 \\
T_a(2) = 3 \\
T_a(3) = 0
\]

Under this identification, \( T_a(n) = T_a(n + 1) \mod 4. \)
For larger powers of 2.

\[
\{0, 1\}^n \iff \mathbb{Z}/2^n\mathbb{Z}
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\[
x_0x_1x_2\ldots x_{n-1} \iff \left(\sum_{i=0}^{n-1} x_i2^i\right) \mod n.
\]

Under this identification \(T_a(x) = (x + 1) \mod 2^n\).
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Under this identification \(T_a(x) = (x + 1) \mod 2^n.\)
The group generated by \( \{ T_a, T_e \} \) is a subgroup of \( \text{Aut}(X^*) \) called \textit{the odometer group}.

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Isomorphic to $\mathbb{Z}$. 
Groups Generated by Automata

Proposition

Let $A = (S, X, P, T)$ be a non-initial Mealy automaton. If for each $s \in S$, the function $T_s : X \to X$ is a permutation of $X$, then the functions $\{T_s\}_{s \in S}$ generate a subgroup of $\text{Aut}(X^*)$.

We usually refer to this subgroup as the group generated by $A$.

We also usually refer to $s$ instead of $T_s$. 
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The Grigorchuk Group

The famed Grigorchuk group is another example of a group generated by an automaton.

“There are other mathematical objects which play a really important role in their subject: the hyperbolic plane for spaces of negative curvature, $\text{SL}(2, \mathbb{R})$ for representations of semi-simple groups, or the hyperfinite $\text{II}_1$ factor for von Neumann algebras...In our opinion, the Grigorchuk group is a good candidate for membership in this club.”

-Pierre de la Harpe,
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Problems Solved By Grigorchuk Group

- Burnside Problem - provide examples of infinite groups which are finitely generated such that each element has finite order.
- First example of group of intermediate growth (answered 1968 question of Milnor)
- First example of amenable but not elementary amenable group. (answered 1957 question of Day)
- Led to introduction of *self-similarity* in group theory.
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\( wX^* \cong X^* \) for any word \( w \).
For each $g \in \text{Aut}(X^*)$ and $w \in X^*$, there exists a unique element $g_w$ such that $g(wv) = g(w)g_w(v)$ for all $v \in X^*$.

This element $g_v$ is called the *section* of $g$ at $v$. 
Self-Similar Groups

Definition

A self-similar group is a subgroup of $\text{Aut}(X^*)$ which contains all sections of all of its elements.

Various properties of self-similar groups have received attention in the last 20 years.
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Various properties of self-similar groups have received attention in the last 20 years.
The remainder of this talk is about the Hausdorff dimension of finitely constrained groups of binary tree automorphisms.

Henceforth, $X = \{0, 1\}$. Thus

$$\text{Sym}(X) \cong C_2 = \{\text{id}, \sigma\} = \{\circ, \bullet\}.$$ 

$G = \text{Aut}(X^*)$. 
Given $g \in G$, the root activity $\alpha : G \to C_2$ is defined by

$$\alpha(g) = \begin{cases} 
\circ, & g(0) = 0 \\
\bullet, & g(0) = 1 
\end{cases}.$$ 

For $w \in X^*$, the activity at $w$ is $\alpha(g_w)$.
$G \iff (C_2)(X^*)$ via the portrait map $[\tau(g)](w) = \alpha(g_w)$

Abusing notation, we write $[\tau(g)](w)$ as $g(w)$. 
We define the group $G(d) = \text{Aut}(X^{[d]})$.

Note that there is a natural homomorphism $\pi_d : G \rightarrow G(d)$.

$$G(d) \cong \text{Sylow } 2 - \text{subgroup of } \text{Sym}(2^d)$$

$$\cong C_2 \lhd C_2 \lhd \ldots \lhd C_2$$

$d$
\( G \) is a profinite (pro-2) topological group as the inverse limit of the inverse system

\[
\ldots \rightarrow G(3) \rightarrow G(2) \rightarrow G(1) \cong C_2.
\]
For $g, h \in G$, we have

$$d(g, h) = \begin{cases} 0, & g = h \\ \frac{1}{2^{2^n - 1}}, & g \neq h \end{cases}$$

where

$$n = \inf\{k \mid \text{there exists } u \in X^k \text{ s.t. } g(u) \neq h(u)\}$$

$$= \inf\{k \mid \pi_k(g) \neq \pi_k(h)\}.$$
Algebra of Continuous Functions on Aut($X^*$)

\[ \mathbb{A} \] - algebra of continuous functions on Aut($X^*$) (pointwise addition and multiplication)

Given $w \in X^*$, we define a function $[w] : \text{Aut}(X^*) \to \mathbb{C}_2$ by $[w](g) = \alpha(gw)$.

Proposition (Siegenthaler, 2009)

Every element of $\mathbb{A}$ is a polynomial in a finite number of variables $[v]$, $v \in X^*$. 
Proposition (Siegenthaler, 2009)

The topological structure of $\text{Aut}(X^*)$ is equivalent to a Zariski topology coming from $\Delta$. 
Hausdorff Dimension

*Hausdorff dimension* - a topological dimension defined for any metric space.

With the metric introduced above, we can naturally associate a Hausdorff dimension to closed subgroups of $G$. 
Hausdorff Dimension of Pro-\(p\) Groups

- Abercrombie, 1994
- Barnea and Shalev, 1997
- Abert and Virag, 2005 – for any \(\lambda \in [0, 1]\), there exists a subgroup of \(\text{Aut}(X^*)\) with Hausdorff dimension \(\lambda\)
Portraits of $G(d)$

$G(d) \iff (C_2)^{X(d)}$ established by portrait map.

$$[\alpha(g)](w) = \begin{cases} \circ, & \text{if } g(w0) = g(w)0 \\ \bullet, & \text{if } g(w0) = g(w)1 \end{cases}$$
Tree Automorphisms and Patterns

Definition

A *pattern* of size $d$ is an element of $G(d)$. A *pattern group of pattern size $d$* is a subgroup of $G(d)$. 

Definition

Given a pattern \( p \in G(d) \) and \( g \in G \), we say \( p \) appears in \( g \) if there exists \( w \in X^* \) such that \( g(wv) = p(v) \) for all \( v \in X(d) \).
Finitely Constrained Groups

Definition
Let $P$ be a proper nontrivial pattern group with pattern size $d$. We define

$$G_P = \{ g \in \text{Aut}(X^*) \mid p \text{ appears in } g \implies p \in P \}.$$ 

We say $G_P$ is defined by $P$.

Definition
Let $H \leq G = \text{Aut}(X^*)$. We say $H$ is a finitely constrained group if there exists $P \preceq G(d)$ such that $H$ is defined by $P$.

Any finitely constrained group is a closed, self-similar subgroup of $G$. 
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Any finitely constrained group is a closed, self-similar subgroup of $G$. 
If $P \leq G(d)$, 
$\dim_H(G_P) \in \{0, \frac{1}{2^{d-1}}, \frac{2}{2^{d-1}}, \ldots, 1 - \frac{2}{2^{d-1}}, 1 - \frac{1}{2^{d-1}}, 1\}$ 
(Bartholdi, 2006; Šunić, 2007)

**Lemma (P, 2014)**

*Given any $d \in \mathbb{N}$ and any $1 \leq k \leq 2^{d-1}$, there exists a finitely constrained group with Hausdorff dimension $1 - \frac{k}{2^{d-1}}$.***
Hausdorff Dimension of Finitely Constrained Groups

If $P \leq G(d)$,
$$\dim_H(G_P) \in \{0, \frac{1}{2^{d-1}}, \frac{2}{2^{d-1}}, \ldots, 1 - \frac{2}{2^{d-1}}, 1 - \frac{1}{2^{d-1}}, 1\}$$
(Bartholdi, 2006; Šunić, 2007)

**Lemma (P, 2014)**

*Given any $d \in \mathbb{N}$ and any $1 \leq k \leq 2^{d-1}$, there exists a finitely constrained group with Hausdorff dimension $1 - \frac{k}{2^{d-1}}$.***
Topological Finite Generation

Definition
A subgroup of $G$ is called *topologically finitely generated* if it is the topological closure of a finitely generated group.

Question
For pattern size $d$ and $1 \leq k \leq 2^{d-1}$, does there exist a *topologically finitely generated*, finitely constrained group of Hausdorff dimension $1 - \frac{k}{2^{d-1}}$?
<table>
<thead>
<tr>
<th>$d$</th>
<th>$1 - \frac{1}{2^{d-1}}$</th>
<th>$1 - \frac{2}{2^{d-1}}$</th>
<th>$1 - \frac{3}{2^{d-1}}$</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Šunić, 2011</td>
<td>N/A</td>
<td>N/A</td>
<td></td>
</tr>
</tbody>
</table>

Note: B - S = Bondarenko and Samoilovych
<table>
<thead>
<tr>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<tbody>
<tr>
<td>$2^{d-1}$</td>
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<td>B-S, 2013</td>
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For $d > 5$, the entries are red and the results are not specified.
<table>
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