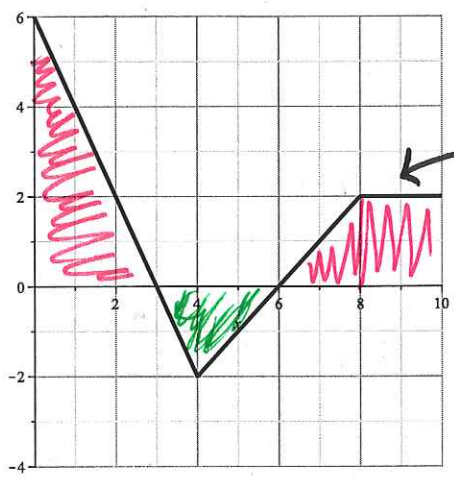
Spring 2012 Math 152

Week in Review I
courtesy: Amy Austin
(covering sections 6.4-6.5)

Section 6.4 and 6.5

1. If $g(x) = \int_0^x f(t)\,dt$, where the graph of $f(t)$ is given below, where $0 \le x \le 10$, evaluate $g(0)$, $g(3)$, $g(6)$ and $g(10)$. What is the maximum value of $g(x)$?



$g(x) = \int_0^x f(t)\,dt$

- $g(0) = \int_0^0 f(t)\,dt = 0$
- $g(3) = \int_0^3 f(t)\,dt = \frac{1}{2}(3)(6) = 9$
- $g(6) = g(3) + \int_3^6 f(t)\,dt = 9 + -\frac{1}{2}(3)(2) = 9 - 3 = 6$
- $g(10) = g(6) + \int_6^{10} f(t)\,dt = 6 + \frac{1}{2}(2)(2) + 4 = 12$
- max value of $g(x) = 12$

2. Find $\frac{d}{dx}\left(\int_{x^2}^{\sin x} \frac{\cos t}{t}\,dt\right)$

$\int_a^b f(x)\,dx = F(x)\Big|_a^b$

$F(x) =$ antideriv. of $f(x)$

$= F(b) - F(a)$

$\frac{d}{dx}\int_{x^2}^{\sin x} \frac{\cos t}{t}\,dt =$

$$\boxed{\frac{\cos(\sin x)}{\sin x} \cdot \cos x - \frac{\cos(x^2)}{x^2} \cdot 2x}$$

$2-\frac{3}{4} \qquad 3-\frac{3}{4}$

3. $\int \frac{\sqrt{x}+x^2-x^3}{\sqrt[4]{x^3}}\,dx =$

$$\int \frac{x^{\frac{1}{2}}+x^2-x^3}{x^{3/4}}\,dx$$

$$\int \left(\frac{x^{\frac{1}{2}}}{x^{\frac{3}{4}}}+\frac{x^2}{x^{3/4}}-\frac{x^3}{x^{3/4}}\right)dx$$

$$=\int \left(x^{-\frac{1}{4}}+x^{\frac{5}{4}}-x^{\frac{9}{4}}\right)dx$$

$$\frac{x^{\frac{3}{4}}}{\frac{3}{4}}+\frac{x^{\frac{9}{4}}}{\frac{9}{4}}-\frac{x^{\frac{13}{4}}}{\frac{13}{4}}+C$$

$$\boxed{\frac{4}{3}x^{\frac{3}{4}}+\frac{4}{9}x^{\frac{9}{4}}-\frac{4}{13}x^{\frac{13}{4}}+C}$$

4. $\int_0^1 (x^3-2)^2\,dx =$

$$\int_0^1 \left(x^6-4x^3+4\right)dx$$

$$\underbrace{\left(\frac{x^7}{7}-\frac{4x^4}{4}+4x\right)}_{F(x)}\Bigg|_0^1$$

$$= F(1)-F(0)$$

$$= \frac{1}{7}-1+4-(0)$$

$$= \frac{1}{7}+3$$

$$= \boxed{\frac{22}{7}}$$

5. $\int \left(\frac{1}{\sqrt{1-x^2}} - 4x^{-1} + 3^x + \frac{2}{x^2+1} - \frac{1}{x^2+4} \right) dx =$

6. $\int 5x^2(3x^3-1)^8\, dx =$

Note: $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$

5. $= \arcsin x - 4\ln|x| + \frac{3^x}{\ln 3} + 2\arctan x - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$

6. $\int 5\cancel{x^2}(\underbrace{3x^3-1}_{u})^8 \underbrace{dx}_{\frac{du}{9\cancel{x^2}}}$

$u = 3x^3 - 1$

$du = 9x^2\, dx$

$\frac{du}{9x^2} = dx$

$\frac{5}{9}\int u^8\, du$

$\frac{5}{9}\,\frac{u^9}{9} + C$

$\boxed{\frac{5}{81}(3x^3-1)^9 + C}$

7. $\int_0^1 x^2 e^{2x^3-5}\,dx =$

$u = 2x^3 - 5$ $\begin{cases} x=1, u=-3 \\ x=0, u=-5 \end{cases}$

$du = 6x^2\,dx$

$\frac{du}{6x^2} = dx$

$\int_{-5}^{-3} \cancel{x^2} e^u \frac{du}{6\cancel{x^2}}$

$\frac{1}{6}\int_{-5}^{-3} e^u\,du$

$\frac{1}{6} e^u \Big|_{-5}^{-3}$

$\boxed{\frac{1}{6}(e^{-3} - e^{-5})}$

8. $\int_{-4}^{0} \frac{1}{\sqrt{1-2x}}\,dx =$

$u = 1-2x$ $\begin{cases} x=0, u=1 \\ x=-4, u=9 \end{cases}$

$du = -2dx$

$\frac{du}{-2} = dx$

$\int_9^1 \frac{1}{\sqrt{u}} \frac{du}{-2}$

$\frac{1}{2}\int_1^9 u^{-\frac{1}{2}}\,du$

$\frac{1}{2} \frac{u^{\frac{1}{2}}}{\frac{1}{2}} \Big|_1^9$

$\sqrt{u}\Big|_1^9 = \sqrt{9} - \sqrt{1}$

$= 3 - 1$

$= \boxed{2}$

9. $\displaystyle\int_1^{1/2} \cos \pi x \, dx =$

$u = \pi x$ $\begin{cases} x = \frac{1}{2}, \; u = \frac{\pi}{2} \\ x = 1, \; u = \pi \end{cases}$

$du = \pi \, dx$

$\frac{du}{\pi} = dx$

$\int_\pi^{\frac{\pi}{2}} \cos(u) \frac{du}{\pi}$

$\frac{1}{\pi} \int_\pi^{\frac{\pi}{2}} \cos u \, du$

$\frac{1}{\pi} \sin u \Big|_\pi^{\frac{\pi}{2}}$

$\frac{1}{\pi}(1-0) = \boxed{\frac{1}{\pi}}$

10. $\displaystyle\int \frac{\overbrace{e^x}^{du}}{\underbrace{1+e^x}_{u}} dx =$

$u = 1 + e^x$

$du = e^x dx$

$\int \frac{du}{u} = \ln|u| + C$

$= \ln|1 + e^x| + C$

what if:

$\int \frac{1+e^x}{e^x} dx$

$\int \left(\frac{1}{e^x} + 1\right) dx$

$\int \left(\underbrace{e^{-x}}_{\text{u-sub}} + 1\right) dx$

$\int e^{-x} dx + \int dx$

$u = -x \qquad du = -dx$

$\int e^u(-du) = -e^u = -e^{-x}$

$\boxed{-e^{-x} + x + C}$

11. $\int \frac{e^x}{1+e^{2x}}\,dx =$

du

$$= \int \frac{e^x}{1+(\underbrace{e^x}_{u})^2}\,dx$$

$u = e^x$

$du = e^x\,dx$

$$= \int \frac{du}{1+u^2}$$

$$= \arctan u + C$$

$$= \boxed{\arctan(e^x) + C}$$

12. $\int_0^{\pi/12} \tan(3x)\,dx =$

~~tan(3x)~~

$$\int_0^{\frac{\pi}{12}} \frac{\sin(3x)\,dx}{\cos(3x)}$$

$\frac{du}{-3}$

u

$u = \cos(3x)$ — $x = \frac{\pi}{12},\ u = \frac{\sqrt{2}}{2}$; $x = 0,\ u = 1$

$du = -3\sin(3x)\,dx$

$\frac{du}{-3} = \sin(3x)\,dx$

$$\int_1^{\frac{\sqrt{2}}{2}} \frac{1}{u}\,\frac{du}{-3}$$

$$-\frac{1}{3}\ln|u| \Big|_1^{\frac{\sqrt{2}}{2}}$$

$$-\frac{1}{3}\left[\ln\left|\frac{\sqrt{2}}{2}\right| - \cancel{\ln(1)}^{\,0}\right]$$

$$\boxed{-\frac{1}{3}\ln\frac{\sqrt{2}}{2}}$$

13. $\int_{-1}^{2} \frac{5}{2x+1}\,dx =$

$u = 2x+1 \begin{cases} x=2, & u=5 \\ x=-1, & u=-1 \end{cases}$

$du = 2\,dx$

$\frac{du}{2} = dx$

$\int_{-1}^{5} \frac{5}{u}\frac{du}{2}$

$\frac{5}{2}\ln|u|\Big|_{-1}^{5}$

$\frac{5}{2}\left[\ln(5) - \ln(1)\right]$

$\boxed{\frac{5}{2}\ln 5}$

Bad problem
improper integral
at $x = -\frac{1}{2}$

14. $\int \frac{\sin t}{\cos^5 t}\,dt =$

$u = \cos t$

$du = -\sin t\,dt$

$-\int \frac{1}{u^5}\,du$

$-\int u^{-5}\,du$

$= -\frac{u^{-4}}{-4} + C$

$= \frac{1}{4u^4} + C$

$= \frac{1}{4\cos^4 t} + C$ or

$= \boxed{\frac{1}{4}\sec^4 t + C}$

15. $\int \frac{x}{\sqrt{x+1}}\,dx =$

$u = x+1 \rightarrow x = u-1$

$du = dx$

$\int \frac{u-1}{\sqrt{u}}\,du$

$\int \left(\frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}}\right) du$

$\int \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right) du$

$\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + C$

$\boxed{\frac{2}{3}(x+1)^{\frac{3}{2}} - 2\sqrt{x+1} + C}$

16. $\int \frac{2x^3}{x^2-1}\,dx =$

$u = x^2 - 1 \longrightarrow x^2 = u+1$

$du = 2x\,dx$

$\frac{du}{2x} = dx$

$= \int \frac{2x^3}{u}\,\frac{du}{2x}$

$= \int \frac{x^2}{u}\,du$

$= \int \frac{u+1}{u}\,du$

$= \int \left(1 + \frac{1}{u}\right) du$

$= u + \ln|u| + C$

$= \boxed{x^2 - 1 + \ln\left|x^2-1\right| + C}$

17. $\int \frac{\sec^2(\sqrt{x})}{\sqrt{x}}\,dx =$ $2du$

$u = \sqrt{x} = x^{\frac{1}{2}}$

$du = \frac{1}{2} x^{-\frac{1}{2}} dx$

$du = \frac{1}{2\sqrt{x}} dx$

$2\,du = \frac{1}{\sqrt{x}} dx$

$= \int (\sec^2 u)(2\,du)$

$= 2\tan u + C$

$= \boxed{2\tan(\sqrt{x}) + C}$

18. $\int \frac{\arctan x}{x^2+1}\,dx =$ du

$u = \arctan x$

$du = \frac{1}{x^2+1} dx$

$= \int u\,du$

$= \frac{u^2}{2} + C$

$= \frac{[\arctan(x)]^2}{2} + C$

Aside: $\int \frac{\arctan(7x)}{1+49x^2} dx$

$u = \arctan(7x)$

$du = \frac{7}{1+49x^2}$

⋮

19. $\int \frac{\cos(\ln x)}{x}\,dx =$ du

$u = \ln x$

$du = \frac{1}{x}\,dx$

$= \int \cos(u)\,du$

$= \sin u + C$

$= \sin(\ln x) + C$

Added problem

$$\int \frac{x+1}{x^2+1}\,dx$$

$$\int \left(\frac{x}{x^2+1} + \frac{1}{x^2+1}\right) dx$$

arctan x

$u = x^2+1$

$du = 2x\,dx$

$= \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u|$

$= \frac{1}{2}\ln|x^2+1|$

$$\boxed{\frac{1}{2}\ln|x^2+1| + \arctan x + C}$$