# Fall 2004 Math 151
# 6 Integrals
# 6.1 Sigma Notation
# Mon, 15/Nov ©2004, Art Belmonte

## *Summary*

### Sigma notation

For integers $m \le n$ and real numbers $a_m, a_{m+1}, \ldots, a_n$, we write

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_n.$$

The left-hand side is shorthand for the finite sum on right. The **index of summation** $k$ takes on integer values from $m$ to $n$. Other indices may be used, such as $i$ and $j$.

### Properties of finite summation

Let $c$ be a constant. Then

- $\sum_{k=m}^{n} c a_k = c \sum_{k=m}^{n} a_k$
- $\sum_{k=m}^{n} (a_k + b_k) = \sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k$
- $\sum_{k=m}^{n} (a_k - b_k) = \sum_{k=m}^{n} a_k - \sum_{k=m}^{n} b_k$

### Particular sums

(a) $\sum_{k=1}^{n} 1 = n$

(b) $\sum_{k=1}^{n} c = cn$

(c) $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

(d) $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

(e) $\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

(f) $\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)\left(3n^2+3n-1\right)}{30}$

## *Hand Examples*

Apply formulas from the Summary where applicable. These problems are essentially exercises in pattern recognition. As such, they are ripe for computer implementation, as we'll see in the MATLAB examples!

### 368/2

Write the sum $\sum_{i=1}^{6} \frac{1}{i+1}$ in expanded form.

### Solution

We have $\sum_{i=1}^{6} \frac{1}{i+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$.

### 368/8

Write the sum $\sum_{j=n}^{n+3} j^2$ in expanded form.

### Solution

We have $\sum_{j=n}^{n+3} j^2 = n^2 + (n+1)^2 + (n+2)^2 + (n+3)^2$.

### 368/10

Write the sum $\sum_{i=1}^{n} f(x_i)\Delta x_i$ in expanded form.

### Solution

We have

$$\sum_{i=1}^{n} f(x_i)\Delta x_i = f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \cdots + f(x_n)\Delta x_n.$$

### 368/14

Write the sum $\frac{3}{7} + \frac{4}{8} + \frac{5}{9} + \frac{6}{10} + \cdots + \frac{23}{27}$ in sigma notation.

### Solution

We have $\sum_{k=3}^{23} \frac{k}{k+4}$.

### 368/18

Write the sum $\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}$ in sigma notation.

### Solution

This is a sum of reciprocals of squares, $\sum_{k=1}^{6}\frac{1}{k^2}$.

### 369/20

Write the sum $1-x+x^2-x^3+\cdots+(-1)^n x^n$ in sigma notation.

### Solution

This is an example of an **alternating sum**, $\sum_{k=0}^{n}(-1)^k x^k$.

### 369/23

Find the value of the sum $\sum_{j=1}^{6}3^{j+1}$.

### Solution

We use brute force.

$$\begin{aligned}\sum_{j=1}^{6}3^{j+1} &= 3^2+3^3+3^4+3^5+3^6+3^7\\ &= 9+27+81+243+729+2187\\ &= 3276\end{aligned}$$

### 369/26

Find the value of the sum $\sum_{i=1}^{100}4$.

### Solution

We have $\sum_{i=1}^{100}4=4\sum_{i=1}^{100}1=4(100)=400$.

### 369/35

Find the value of the sum $\sum_{i=1}^{n}\left(i^3-i-2\right)$.

### Solution

Apply the formulas.

$$\begin{aligned}\sum_{i=1}^{n}\left(i^3-i-2\right) &= \sum_{i=1}^{n}i^3-\sum_{i=1}^{n}i-2\sum_{i=1}^{n}1\\ \text{[Stop; this is fine.]}\rightarrow\quad &= \left(\frac{n(n+1)}{2}\right)^2-\frac{n(n+1)}{2}-2n\\ &= \frac{1}{4}\left(n^4+2n^3+n^2\right)-\frac{1}{2}\left(n^2+n\right)-2n\\ \text{[Compare with MATLAB.]}\rightarrow\quad &= \frac{1}{4}n^4+\frac{1}{2}n^3-\frac{1}{4}n^2-\frac{5}{2}n\end{aligned}$$

### 369/45

Find the limit $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\left(\left(\frac{2i}{n}\right)^3+5\left(\frac{2i}{n}\right)\right)$.

### Solution

- First compute the sum. Note that the index of summation is $i$. The letter $n$ is a *fixed* positive integer. It is $i$ that varies!

$$\begin{aligned}&\sum_{i=1}^{n}\frac{2}{n}\left(\left(\frac{2i}{n}\right)^3+5\left(\frac{2i}{n}\right)\right)\\ =\ &\frac{2}{n}\sum_{i=1}^{n}\left(\frac{2^3}{n^3}i^3+\frac{10}{n}i\right)\\ =\ &\frac{2}{n}\left(\frac{2^3}{n^3}\sum_{i=1}^{n}i^3+\frac{10}{n}\sum_{i=1}^{n}i\right)\\ =\ &\frac{2}{n}\left(\frac{2^3}{n^3}\left(\frac{n(n+1)}{2}\right)^2+\frac{10}{n}\left(\frac{n(n+1)}{2}\right)\right)\\ =\ &\frac{2}{n}\left(\frac{2}{n^3}\left(n^4+2n^3+n^2\right)+5(n+1)\right)\\ =\ &\frac{2}{n}\left(2\left(n+2+\frac{1}{n}\right)+5n+5\right)\\ =\ &\frac{2}{n}\left(7n+9+\frac{2}{n}\right)\\ =\ &14+\frac{18}{n}+\frac{4}{n^2}\end{aligned}$$

- *Now* take the limit: $\lim_{n\to\infty}\left(14+\frac{18}{n}+\frac{4}{n^2}\right)=14$.
- Therefore, $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\left(\left(\frac{2i}{n}\right)^3+5\left(\frac{2i}{n}\right)\right)=14$.

If that wasn't a world 'o' hurt, I don't know what is! This is why we we use computers: they are better at pattern recognition than you are. See the corresponding MATLAB example.

## 369/41(c)

Evaluate the **telescoping sum** $\displaystyle\sum_{i=3}^{99}\left(\frac{1}{i}-\frac{1}{i+1}\right)$.

### Solution

The nomenclature means that the sum "collapses."

$$\begin{aligned}\textstyle\sum_{i=3}^{99}\left(\frac{1}{i}-\frac{1}{i+1}\right) &= \left(\tfrac{1}{3}-\tfrac{1}{4}\right)+\left(\tfrac{1}{4}-\tfrac{1}{5}\right)+\cdots\left(\tfrac{1}{98}-\tfrac{1}{99}\right)+\left(\tfrac{1}{99}-\tfrac{1}{100}\right)\\ &= \frac{1}{3}-\frac{1}{100}=\frac{100-3}{300}=\frac{97}{300}\end{aligned}$$

## *MATLAB Examples*

### s369x35 [revisited]

Find the value of the sum $\displaystyle\sum_{i=1}^{n}\left(i^3-i-2\right)$.

### Solution

The MATLAB command **symsum** (symbolic summation) makes quick work of this one! The answer agrees with the one we obtained by hand.

```
%-------------------------------------------------
% Stewart 369/35
%
syms i n
our_sum = simplify(symsum(i^3 - i - 2,  i,  1,  n));
pretty(our_sum)

                          4         3         2
                 1/4 n  + 1/2 n  - 1/4 n  - 5/2 n
%
echo off; diary off
```

### s369x45 [revisited]

Find the limit $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\left(\left(\frac{2i}{n}\right)^3+5\left(\frac{2i}{n}\right)\right)$.

### Solution

A half page of hand symbolic manipulation is reduced to one line of code. "Can you say power tool?" I knew you could. . .

```
%-------------------------------------------------
% Stewart 369/45
%
syms i n
S = symsum(2/n * ( (2*i/n)^3 + 5*(2*i/n) ),  i,  1,  n);
S = expand(S); pretty(S)

                                          18     4
                                   14 + ---- + ----
                                          n       2
                                                 n
L = limit(S, n, inf)

L =

14


%
echo off; diary off
```

### s369x47

Prove the formula for the sum of a **finite geometric series** with first term $a$ and common ratio $r$.

$$\sum_{i=1}^{n}ar^{i-1}=a+ar+ar^2+\cdots+ar^{n-1}=\frac{a\left(r^n-1\right)}{r-1}$$

### Solution

```
%-------------------------------------------------
% Stewart 369/47: Sum of a finite geometric series
%
syms a i n r
GS = symsum(a * r^(i-1),  i,  1,  n);
GS = simplify(GS); pretty(GS)

                                          n
                                     a (r  - 1)
                                     ----------
                                       r - 1
%
echo off; diary off
```

### s369x48

Evaluate $\displaystyle\sum_{i=1}^{n}\frac{3}{2^{i-1}}$.

### Solution

Rewrite the sum as $\displaystyle\sum_{i=1}^{n}3\left(\tfrac{1}{2}\right)^{i-1}$, a finite geometric series with $a=3$ and $r=\frac{1}{2}$. Then apply the result from the preceding problem to obtain the sum $\dfrac{3\left(\left(\frac{1}{2}\right)^n-1\right)}{\frac{1}{2}-1}=6\left(1-2^{-n}\right)$.

```
%-------------------------------------------------
% Stewart 369/48: Sum of a PARTICULAR finite geometric series
%
syms i n
S = simplify( symsum(3 / 2^(i-1),  i,  1,  n) ); pretty(S)

                                         (-n)
                                     -6 2     + 6
%
echo off; diary off
```

### s369x50

Evaluate $\sum_{i=1}^{m}\left(\sum_{j=1}^{n}(i+j)\right)$.

### Solution

This is a *double* finite sum.

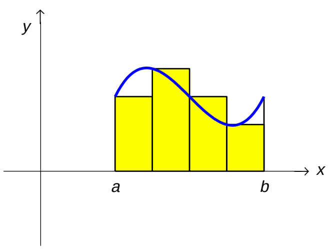
```
%--------------------------------------------------
% Stewart 369/50: A finite DOUBLE sum
%
syms i j m n
S = symsum( symsum(i+j, j, 1, n), i, 1, m );
S = simplify(S); pretty(S)

                                 2            2
                           1/2 m  n + 1/2 m n  + m n
%
echo off; diary off
```

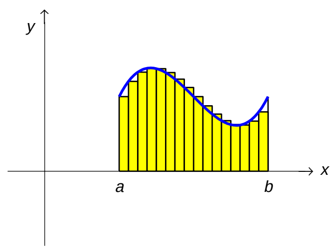
# Fall 2004 Math 151
# 6 Integrals
## 6.2 Area
**Mon, 15/Nov**



### *Summary*

Let $f$ be a function defined on $I = [a, b]$ with $f \geq 0$ on $I$. We seek the area of the region $R$ bounded above by the curve $y = f(x)$, below by the $x$-axis, on the left by the vertical line $x = a$, and on the right by the vertical line $x = b$. Approximate the area by adding up the areas of rectangular strips as follows.




Split the interval $[a, b]$ into $n$ subintervals whose endpoints constitute a **partition**

$$P : a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Let $x_i^* \in [x_{i-1}, x_i]$ be in the $i$th subinterval and $\Delta x_i = x_i - x_{i-1}$ be the length of this subinterval. We define the **norm** of $P$ by $\|P\| = \max_{1 \leq i \leq n} \Delta x_i$. Now let the number of subintervals $n$ increase indefinitely while the norm of $P$ shrinks to 0. The **area** $A$ of $R$ is

$$A = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*) \, \Delta x_i,$$

provided that this limit of the sum of the areas of the rectangles formed by the partitions exists.

### *Hand Examples*

Apply formulas from the Section 6.1 Summary when necessary.

#### 377/3

Let $f(x) = 16 - x^2$, $[a, b] = [0, 4]$, $P = \{0, 1, 2, 3, 4\}$, and $x_i^* =$ midpoint.

(a) Find $\|P\|$, the norm of $P$.

(b) Find $\sum_{i=1}^{n} f(x_i^*) \, \Delta x_i$, the sum of the areas of approximating rectangles as given in the Summary.

(c) Sketch the graph of $f$ and the approximating rectangles.

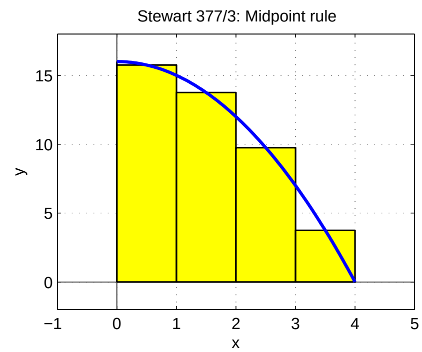
#### Solution

(a) We have $\|P\| = \max\{1, 1, 1, 1\} = 1$, the length of the longest subinterval in the partition.

(b) The sum of the areas of approximating rectangles is

$$\begin{aligned} & \sum_{i=1}^{4} f(x_i^*) \, \Delta x_i \\ = \; & (15.75)(1) + (13.75)(1) + (9.75)(1) + (3.75)(1) \\ = \; & 43. \end{aligned}$$

(c) Here is a graph of $f$ and the approximating rectangles.




#### 377/8

Let $f(x) = 4 \cos x$, $[a, b] = [0, \frac{\pi}{2}]$, $P = \{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\}$, and $x_i^* =$ left endpoint.

(a) Find $\|P\|$, the norm of $P$.

(b) Find $\sum_{i=1}^{n} f(x_i^*) \, \Delta x_i$, the sum of the areas of approximating rectangles as given in the Summary.

(c) Sketch the graph of $f$ and the approximating rectangles.

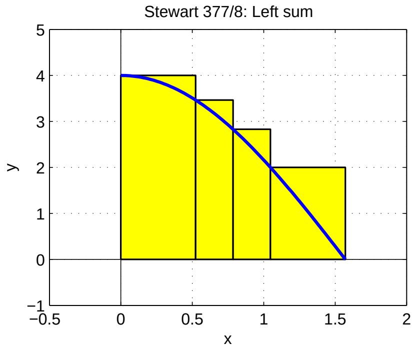
#### Solution

(a) We have $\|P\| = \max\{\frac{\pi}{6}, \frac{\pi}{12}, \frac{\pi}{12}, \frac{\pi}{6}\} = \frac{\pi}{6}$, the length of the longest subinterval in the partition.

(b) The sum of the areas of approximating rectangles is

$$\begin{aligned} & \sum_{i=1}^{4} f(x_i^*) \, \Delta x_i \\ = \; & (4)\left(\tfrac{\pi}{6}\right) + \left(2\sqrt{3}\right)\left(\tfrac{\pi}{12}\right) + \left(2\sqrt{2}\right)\left(\tfrac{\pi}{12}\right) + (2)\left(\tfrac{\pi}{6}\right) \\ = \; & \tfrac{1}{6}\left(\sqrt{3} + \sqrt{2} + 6\right)\pi \approx 4.789. \end{aligned}$$

(c) Here is a graph of $f$ and the approximating rectangles.

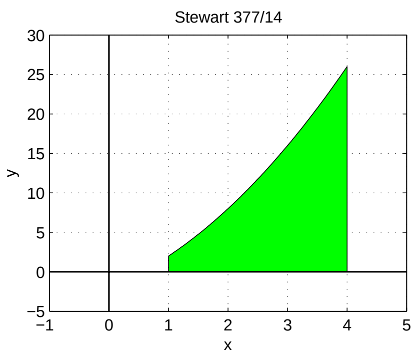



### 377/14

Find the exact area under the curve $y = f(x) = x^2 + 3x - 2$ and above the $x$-axis between $a = 1$ and $b = 4$. Use *equal* subintervals and $x_k$ to be the right endpoint of the $k$th subinterval. Also sketch the region. (NOTE: For brevity, we'll write $\sum$ for $\sum_{k=1}^{n}$).

### Solution

- Here is a sketch of the region whose area we seek.




- The length of each subinterval is

$$\Delta x_k = \Delta x = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n}$$

whereas the right endpoint of the $kth$ subinterval is

$$x_k^* = a + k\Delta x = 1 + \frac{3k}{n}.$$

- The sum of the areas of the approximating rectangles is

$$\begin{aligned}
\sum f(x_k^*)\Delta x &= \Delta x \sum f(x_k^*) \quad \text{[since } \Delta x \text{ is constant]} \\
&= \tfrac{3}{n} \sum \left( (1 + \tfrac{3k}{n})^2 + 3(1 + \tfrac{3k}{n}) - 2 \right) \\
&= \tfrac{3}{n} \sum \left( 1 + \tfrac{6}{n}k + \tfrac{9}{n^2}k^2 + 3 + \tfrac{9}{n}k - 2 \right) \\
&= \tfrac{3}{n} \left( (\sum 2) + (\tfrac{15}{n} \sum k) + (\tfrac{9}{n^2} \sum k^2) \right) \\
&= \tfrac{3}{n} \left( 2n + \tfrac{15n(n+1)}{2n} + \tfrac{9n(n+1)(2n+1)}{6n^2} \right)
\end{aligned}$$

$$= \quad 6 + \frac{45}{2}\left(1 + \frac{1}{n}\right) + \frac{9}{2}(1)\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) = S_n$$

- Now let $n \to \infty$ to obtain

$$A = \lim_{n\to\infty} S_n = 6 + \frac{45}{2} + 9 = 15 + \frac{45}{2} = \frac{75}{2} = 37.5.$$

The area is 37.5 square units. (Also see MATLAB example.)

## *MATLAB Examples*

### s377x10

Let $f(x) = 4x - x^3$.

(a) Sketch the region that lines under the curve $y = f(x)$ above the $x$-axis from $x = 0$ to $x = 2$.

(b) Find an expression for $R_n$, the sum of the areas of the $n$ approximating rectangles, taking $x_k^*$ to be the right endpoint and using subintervals of equal length.

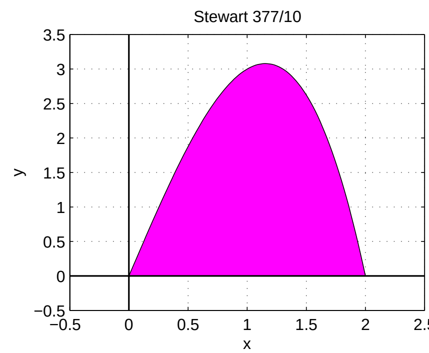
(c) Find the numerical values of the approximating areas $R_n$ for $n = 10, 20, 30$.

(d) Find the exact area of the region.

### Solution

A diary file at the end shows all computations and plot commands.

(a) Here is a sketch of the region whose area we seek.




(b) We have $R_n = 4 - \dfrac{4}{n^2}$.

(c) Here are values of $R_n$ for the requested values of $n$.

| $n$ | 10 | 20 | 30 |
|---|---|---|---|
| $R_n$ | 3.96 | 3.99 | 3.996 |

(d) The exact area is $\lim\limits_{n\to\infty} R_n = \lim\limits_{n\to\infty} \left(4 - \dfrac{4}{n^2}\right) = 4.$

```
%---------------------------------------------------
% Stewart 377/10: Area of region as limit of sum
% of areas of approximating rectangles
%
% (b)
syms k n x
f = inline('4.*x - x.^3', 'x');
a = 0; b = 2; dx = (b-a)/n; % dx = step size
xk = a + k*dx; % right endpoint of kth subinterval
Rn = symsum(f(xk)*dx, k, 1, n); % right sum
Rn = expand(simplify(Rn)); pretty(Rn)

                                     4
                                4 - ----
                                      2
                                     n
% (c)
N = [10 20 30]; RSN = [];
for m = N
    RSN = [RSN subs(Rn, n, m)];
    echo off
end
n  Rn
10 3.960000
```

```
20 3.990000
30 3.995556
% echo on
% (d)
A = limit(Rn, n, inf) % area

A =

4


%
echo off; diary off
%-------------------------------------------------
% Stewart 377/10g: Sketch of region
%
f = inline('4.*x - x.^3', 'x');
x = linspace(0, 2);
y = f(x);
xf = [x 2 0]; yf = [y 0 0];
fill(xf,yf, 'm'); hold on
plot([-0.5 2.5], [0 0], 'k', 'LineWidth', 1)
plot([0 0], [-0.5 3.5], 'k', 'LineWidth', 1)
grid on; axis([-0.5 2.5 -0.5 3.5])
xlabel('x'); ylabel('y'); title('Stewart 377/10')
%
echo off; diary off
```

## s377x14 [revisited]

Find the exact area under the curve $y = f(x) = x^2 + 3x - 2$ and above the $x$-axis between $a = 1$ and $b = 4$. Use *equal* subintervals and $x_k$ to be the right endpoint of the $k$th subinterval.

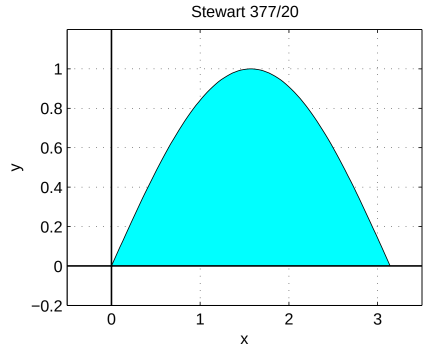
## Solution

MATLAB's **symsum** command rapidly yields the needful.

```
%-------------------------------------------------
% Stewart 377/14: Area of region as limit of sum
% of areas of approximating rectangles
%
syms k n x
f = inline('x.^2 + 3.*x - 2', 'x');
a = 1; b = 4; dx = (b-a)/n; % dx = step size
xk = a + k*dx; % right endpoint of kth subinterval
RS = symsum(f(xk)*dx, k, 1, n); % right sum
RS = expand(simplify(RS)); pretty(RS)

                              36         1
                    75/2 + ---- + 9/2 ----
                              n          2
                                        n
A = limit(RS, n, inf) % area

A =

75/2


%
echo off; diary off
```

## s377x20

Consider the region below the curve $y = f(x) = \sin x$ above the $x$-axis between $x = 0$ and $x = \pi$. Compute the sum of the areas of approximating rectangles using equal subintervals and right endpoints for $n = 10, 30, 50$. Guess the exact value of the area.




## Solution

- Here are values of $R_n$ for the requested values of $n$.

| $n$ | 10 | 30 | 50 |
|---|---|---|---|
| $R_n$ | 1.983524 | 1.998172 | 1.999342 |

- The exact area is $\lim_{n\to\infty} R_n = 2$.

```
%-------------------------------------------------
% Stewart 377/20: Area of region as limit of sum
% of areas of approximating rectangles
%
% (b)
syms k n x
f = inline('sin(x)', 'x');
a = 0; b = pi; dx = (b-a)/n; % dx = step size
xk = a + k*dx; % right endpoint of kth subinterval
Rn = symsum(f(xk)*dx, k, 1, n); % right sum
Rn = expand(simplify(Rn)); pretty(Rn)

                                       pi
                                pi sin(----)
                                        n
                           - -----------------
                                /     pi      \
                              n |cos(----) - 1|
                                \     n       /
% (c)
N = [10 30 50]; RSN = [];
for m = N
    RSN = [RSN subs(Rn, n, m)];
    echo off
end
n  Rn
10 1.983524
30 1.998172
50 1.999342
% echo on
% (d)
A = limit(Rn, n, inf) % area

A =

2


%
echo off; diary off
```

# Fall 2004 Math 151
# 6 Integrals
# 6.3 The Definite Integral
# Fri, 19/Nov ©2004, Art Belmonte

## *Summary*

### Definitions

Let $f$ be a function defined on $I = [a, b]$. (NOTE: In this section we remove the restriction that $f \geq 0$ on $I$.) Split the interval $[a, b]$ into $n$ subintervals whose endpoints constitute a **partition**

$$P : a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

(Often the $x_i$ are equally spaced and we have a **regular partition**.) Let $x_i^* \in \left[x_{i-1}, x_i\right]$ be in the $i$th subinterval and $\Delta x_i = x_i - x_{i-1}$ be the length of this subinterval. Recall that the norm of $P$ is defined by $\|P\| = \max \Delta x_i$. Now let the number of subintervals $n$ increase indefinitely while the norm of $P$ shrinks to 0. The **definite integral of $f$ from $a$ to $b$** is defined by

$$\int_a^b f(x)\,dx = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f\left(x_i^*\right) \Delta x_i,$$

provided the limit exists. When this occurs, $f$ is said to be **integrable** on $[a, b]$. Here are some terms.

- The process of computing the value of an integral is called **integration**.
- The symbol $\int$ is called an **integral sign**. It was introduced by Leibniz, one of the inventors of Calculus in the 1680s.
- The function $f(x)$ is the **integrand**.
- The numbers $a$ and $b$ are called **limits of integration**; $a$ is the **lower limit** and $b$ the **upper limit**.
- The sums $\sum_{i=1}^{n} f\left(x_i^*\right) \Delta x_i$ are called **Riemann sums**.
- The integral defined above is known as the **Riemann integral**.

### Sufficient conditions for a definite integral to exist

If any *one* of these conditions holds, then $f$ is integrable on $[a, b]$.

- $f$ is *continuous* on $[a, b]$.
- $f$ is *piecewise continuous* on $[a, b]$; i.e., $f$ is continuous on $[a, b]$ except for a finite number of jump discontinuities.
- $f$ is *monotonic* on $[a, b]$; i.e., increasing on $[a, b]$ or decreasing on $[a, b]$.

### Properties of the definite integral

Let $c$, $m$, and $M$ be constants and let $f$ and $g$ be integrable on $[a, b]$, where $a \leq b$. Then the following properties hold.

1. $\int_a^b c\,dx = c(b - a)$
2. $\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
3. $\int_a^b f(x) - g(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$
4. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
5. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
6. If $f \geq 0$ on $[a, b]$, then $\int_a^b f(x)\,dx \geq 0$. In this case, the integral represents the area under the curve $y = f(x)$ and above the $x$-axis between $x = a$ and $x = b$, as in Section 6.2.
7. If $f \leq g$ on $[a, b]$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.
8. If $m \leq f \leq M$ on $[a, b]$, then
$$m\,(b - a) \leq \int_a^b f(x)\,dx \leq M\,(b - a)\,.$$
9. $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$.

### Rules for approximating the definite integral $\int_a^b f(x)\,dx$

These are *very easy* to implement in MATLAB. Each uses regular partitions with equal step size (subinterval length) $h = \frac{1}{n}(b - a)$. The Midpoint Rule is the most accurate of these three rules.

- **Left sum rule**: $L_n = h\sum_{k=0}^{n-1} f(a + kh)$
- **Right sum rule**: $R_n = h\sum_{k=1}^{n} f(a + kh)$
- **Midpoint Rule**: $M_n = h\sum_{k=0}^{n-1} f(a + (k + \frac{1}{2})h)$

### Miscellaneous definitions

- If $a = b$, then $\int_a^b f(x)\,dx = \int_a^a f(x)\,dx = 0$.
- If $a > b$, then $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$, provided the latter exists as a limit.

## *Hand Examples*

Apply formulas from the Section 6.1 Summary when necessary.

### 386/9

Use the Midpoint Rule with $n = 5$ to approximate $\displaystyle\int_0^5 x^3\,dx$.

### Solution

Here $a = 0$ and $b = 5$. We have $h = \frac{1}{n}(b-a) = \frac{1}{5}(5-0) = 1$ and $a + (k+\frac{1}{2})h = k + \frac{1}{2}$, $k = 0, 1, 2, 3, 4$. Now $f(x) = x^3$, so

$$\begin{aligned}
\text{Midpoint Rule: } M_n &= h\sum_{k=0}^{n-1} f(a+(k+\tfrac{1}{2})h) \\
&= \sum_{k=0}^{4} f(k+\tfrac{1}{2}) \\
&= f(\tfrac{1}{2}) + f(\tfrac{3}{2}) + f(\tfrac{5}{2}) + f(\tfrac{7}{2}) + f(\tfrac{9}{2}) \\
&= \tfrac{1}{8} + \tfrac{27}{8} + \tfrac{125}{8} + \tfrac{343}{8} + \tfrac{729}{8} \\
&= \frac{1225}{8} = 153\tfrac{1}{8} = 153.125.
\end{aligned}$$

### 387/16

Compute $\int_{-2}^{7} 6 - 2x\,dx$ by taking the limit of right sums.

### Solution

Here $a = -2$ and $b = 7$, $h = \frac{1}{n}(b-a) = \frac{1}{n}(7-(-2)) = \frac{9}{n}$ and $a + kh = -2 + \frac{9k}{n}$, $k = 1, \ldots, n$. Now $f(x) = 6 - 2x$, so

$$\begin{aligned}
\text{Right sum rule: } R_n &= h\sum_{k=1}^{n} f(a+kh) \\
&= \frac{9}{n}\sum_{k=1}^{n} f\left(\frac{9k}{n} - 2\right) \\
&= \frac{9}{n}\sum_{k=1}^{n}\left(6 - 2\left(\frac{9k}{n} - 2\right)\right) \\
&= \frac{9}{n}\sum_{k=1}^{n}\left(10 - \frac{18}{n}k\right) \\
&= \frac{9}{n}\left(\left(\sum_{k=1}^{n} 10\right) - \left(\frac{18}{n}\sum_{k=1}^{n} k\right)\right) \\
&= \frac{9}{n}\left(10n - \frac{18}{n}\frac{n(n+1)}{2}\right) \\
&= \frac{9}{n}(n-9) = 9 - \frac{9}{n}
\end{aligned}$$

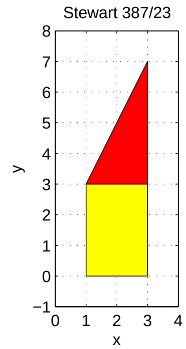
As $n \to \infty$, we have $R_n = 9 - \frac{9}{n} \to 9$. Thus $\int_{-2}^{7} 6 - 2x\,dx = 9$.

### 387/23

Evaluate $\int_{1}^{3} 1 + 2x\,dx$ by interpreting it in terms of areas.

### Solution

From a sketch we see that the value of the integral is the sum of the areas of a rectangle and a triangle: $(2 \times 3) + \frac{1}{2}(2 \times 4) = 10$.

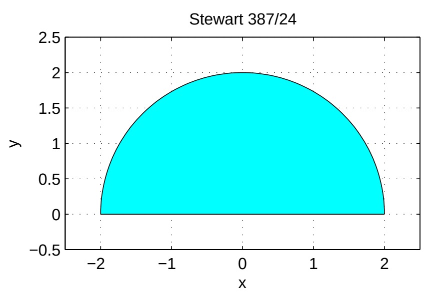



### 387/24

Evaluate $\int_{-2}^{2} \sqrt{4 - x^2}\,dx$ by interpreting it in terms of area.

### Solution

From a sketch we see that the value of the integral is the area of a semicircular region: $\frac{1}{2}\pi\,(2)^2 = 2\pi$.

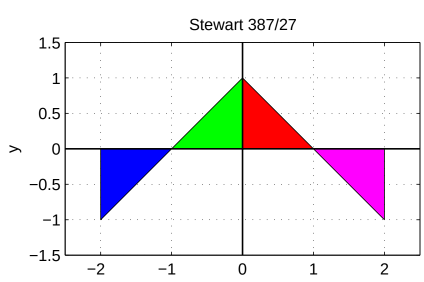



### 387/27

Evaluate $\int_{-2}^{2} 1 - |x|\,dx$ by interpreting it in terms of areas.

### Solution

From a sketch we interpret the integral to be the sum of signed areas. The positive areas above the $x$-axis exactly cancel out the negative areas below the $x$-axis. Hence $\int_{-2}^{2} 1 - |x|\,dx = 0$.

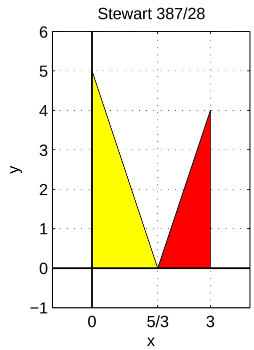


### 387/28

Evaluate $\int_0^3 |3x-5|\ dx$ by interpreting it in terms of areas.

### Solution

From a sketch we see that the value of the integral is the sum of the areas of two triangles: $\frac{1}{2}\left(\frac{5}{3}\times 5\right)+\frac{1}{2}\left(\frac{4}{3}\times 4\right)=\frac{41}{6}=6\frac{5}{6}$.




### 387/34

Express the limit $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+(i/n)^2}$ as a definite integral.

### Solution

Examine the pieces and flesh them out a little.

$$\lim_{n\to\infty}\frac{1-0}{n}\sum_{i=1}^{n}\frac{1}{1+\left(0+\frac{1-0}{n}\,i\right)^2}$$

We recognize this as the limit of the right sums of the integral

$$\int_0^1 \frac{1}{1+x^2}\,dx.$$

### 387/44

Use the properties of the definite integral to evaluate

$$\int_3^4 f(x)\,dx+\int_1^3 f(x)\,dx+\int_4^1 f(x)\,dx.$$

### Solution

Rearrange and combine.

$$\begin{aligned}
&\left(\int_1^3 f(x)\,dx+\int_3^4 f(x)\,dx\right)+\int_4^1 f(x)\,dx\\
=&\left(\int_1^4 f(x)\,dx\right)+\int_4^1 f(x)\,dx\\
=&\int_1^4 f(x)\,dx-\int_1^4 f(x)\,dx=0
\end{aligned}$$

### 387/46

Write the sum $\int_5^8 f(x)\,dx+\int_0^5 f(x)\,dx$ as a single integral.

### Solution

Swap and combine.

$$\int_0^5 f(x)\,dx+\int_5^8 f(x)\,dx=\int_0^8 f(x)\,dx$$

### 387/48

Write the combination

$$\int_{-3}^5 f(x)\,dx-\int_{-3}^0 f(x)\,dx+\int_5^6 f(x)\,dx$$

as a single integral.

### Solution

Rearrange and combine.

$$\begin{aligned}
&\left(\int_{-3}^5 f(x)\,dx+\int_0^{-3} f(x)\,dx\right)+\int_5^6 f(x)\,dx\\
=&\left(\int_0^{-3} f(x)\,dx+\int_{-3}^5 f(x)\,dx\right)+\int_5^6 f(x)\,dx\\
=&\int_0^5 f(x)\,dx+\int_5^6 f(x)\,dx\\
=&\int_0^6 f(x)\,dx
\end{aligned}$$

### 387/52

Use the properties of the definite integral to verify the inequality

$$\frac{\pi}{6}\le\int_{\pi/6}^{\pi/2}\sin x\,dx\le\frac{\pi}{3}$$

without evaluating the integral.

### Solution

For $\frac{\pi}{6}\le x\le\frac{\pi}{2}$, we have $m=\frac{1}{2}\le\sin x\le 1=M$. Applying Property 8 yields

$$\frac{1}{2}\left(\frac{\pi}{2}-\frac{\pi}{6}\right)\le\int_{\pi/6}^{\pi/2}\sin x\,dx\le 1\left(\frac{\pi}{2}-\frac{\pi}{6}\right)$$

or $\frac{\pi}{6}\le\int_{\pi/6}^{\pi/2}\sin x\,dx\le\frac{\pi}{3}$.

### 388/56

Use Property 8 to estimate $\int_0^2 \sqrt{x^3+1}\,dx$.

### Solution

For $0 \le x \le 2$, we have $m = 1 \le \sqrt{x^3+1} \le 3 = M$. Applying Property 8 yields

$$1\,(2-0) \le \int_0^2 \sqrt{x^3+1}\,dx \le 3\,(2-0)$$

or $2 \le \int_0^2 \sqrt{x^3+1}\,dx \le 6$.

### 388/61

Use the properties of the definite integral, together with Exercise 22 (see MATLAB examples below), to prove the inequality

$$\frac{26}{3} \le \int_1^3 \sqrt{x^4+1}\,dx.$$

### Solution

On the interval $[1, 3]$, we have $x^2 \le \sqrt{x^4+1}$, whence

$$\begin{aligned}
\int_1^3 x^2\,dx &\le \int_1^3 \sqrt{x^4+1}\,dx && \text{[Property 7]}\\
\frac{3^3-1^3}{3} &\le \int_1^3 \sqrt{x^4+1}\,dx && \text{[Exercise 22]}\\
\frac{26}{3} &\le \int_1^3 \sqrt{x^4+1}\,dx.
\end{aligned}$$

## *MATLAB Examples*

### s386x08

The table below gives values of a function obtained from an experiment. Use them to estimate $\int_0^6 f(x)\,dx$ using three equal subintervals with (a) right endpoints, (b) left endpoints, and (c) midpoints. SUPPLEMENT: If the function $f$ is known to be a decreasing function, can you say whether your estimates are less than or greater than the exact value of the integral?

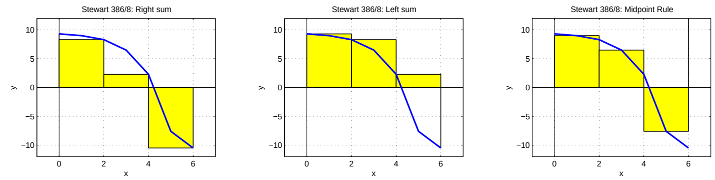
| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 9.3 | 9.0 | 8.3 | 6.5 | 2.3 | $-7.6$ | $-10.5$ |

### Solution

Here are right, left, and middle sums. (A diary file at the end shows all computations.)

| $R_3$ | $L_3$ | $M_3$ |
|---|---|---|
| 0.2 | 39.8 | 15.8 |

The right sum underestimates the integral, the left sum overestimates it, and the middle sum gives the best estimate.




```
%------------------------------------------------
% Stewart 386/8
%
%
a = 0; b = 6; n = 3;
x = 0 : 2 : 6
x =
     0     2     4     6
dx = diff(x)
dx =
     2     2     2
%
yR = [8.3 2.3 -10.5];
yL = [9.3 8.3 2.3];
yM = [9.0 6.5 -7.6];
%
Rn = yR * dx'
Rn =
    0.2000
Ln = yL * dx'
Ln =
   39.8000
Mn = yM * dx'
Mn =
   15.8000
%
echo off; diary off
```

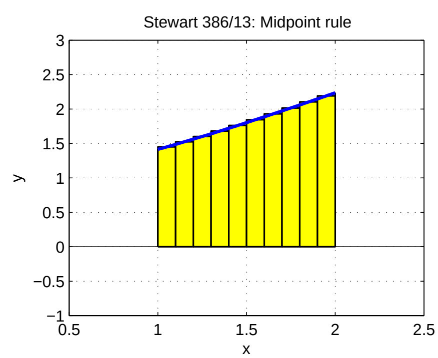
### s386x13

Use the Midpoint Rule with $n = 10, 20, 30$ to approximate $\int_1^2 \sqrt{1+x^2}\,dx$. Illustrate the case for $n = 10$.

### Solution

Here are the middle sums and a plot, followed by a diary file.

| $M_{10}$ | $M_{20}$ | $M_{30}$ |
|---|---|---|
| 1.81001414 | 1.81007263 | 1.81008347 |




```
%------------------------------------------------
% Stewart 386/13: Midpoint rule (n = 10, 20, 30)
%
format long
%
a = 1; b = 2; f = inline('sqrt(1 + x.^2)', 'x');
plot([0.5 2.5], [0 0], 'k') % x-axis
grid on; hold on
```

```
plot([0 0], [-1 3], 'k') % y-axis
axis([0.5 2.5 -1 3])
xlabel('x'); ylabel('y')
title('Stewart 386/13: Midpoint rule')
%
n = 10;
x = linspace(a, b, n+1); % 10 subintervals, 11 part pts
dx = diff(x); % lengths of subintervals!
y = f(x(1:n) + 0.5*(b-a)/n); ys = y;
% Riemann sum
S = y * dx' % midpoint rule
S =
   1.81001414161572
plot([x(1) x(1)], [0 f(x(1))], 'k', 'LineWidth', 1)
plot([x(n+1) x(n+1)], [0 f(x(n+1))], 'k', 'LineWidth', 1)
for k = 1:n % midpoint func vals
    fill([x(k), x(k+1), x(k+1), x(k)], ...
         [0, 0, y(k), y(k)], 'y', 'LineWidth', 1)
    echo off
end
% echo on
%
x = linspace(a, b);
y = f(x);
plot(x,y, 'LineWidth', 2);
%
n = 20;
x = linspace(a, b, n+1); % 20 subintervals, 21 part pts
dx = diff(x); % lengths of subintervals!
y = f(x(1:n) + 0.5*(b-a)/n); ys = y;
% Riemann sum
S = y * dx' % midpoint rule
S =
   1.81007263106288
%
n = 30;
x = linspace(a, b, n+1); % 30 subintervals, 31 part pts
dx = diff(x); % lengths of subintervals!
y = f(x(1:n) + 0.5*(b-a)/n); ys = y;
% Riemann sum
S = y * dx' % midpoint rule
S =
   1.81008346878631
%
format short

echo off; diary off
```

## s387x21

Prove that $\int_a^b x\,dx = \dfrac{b^2 - a^2}{2}$.

## Solution

```
%-------------------------------------------------
% Stewart 387/21
%
syms a b k n x
f = inline('x', 'x');
dx = (b-a)/n; % dx = step size
xk = a + k*dx; % right endpoint of kth subinterval
RS = symsum(f(xk)*dx, k, 1, n); % right sum
RS = expand(simplify(RS)); pretty(RS)

                           2                   2
     a b         2        a          2        b
   - --- - 1/2 a  + 1/2 ---- + 1/2 b  + 1/2 ----
      n                   n                   n
I = limit(RS, n, inf); pretty(I)

                                            2        2
                               - 1/2 a  + 1/2 b
%

echo off; diary off
```

## s387x22

Prove that $\int_a^b x^2\,dx = \dfrac{b^3 - a^3}{3}$.

## Solution

```
%-------------------------------------------------
% Stewart 387/22
%
syms a b k n x
f = inline('x.^2', 'x');
dx = (b-a)/n; % dx = step size
xk = a + k*dx; % right endpoint of kth subinterval
RS = symsum(f(xk)*dx, k, 1, n); % right sum
RS = expand(simplify(RS)); pretty(RS)

           2            2        2
          a  b       a b        a  b
  - 1/2 ---- - 1/2 ---- + 1/2 ----
           n           2          2
                      n          n
                                3          3
          3        3           b          b
 - 1/3 a  + 1/3 b  + 1/2 ---- + 1/6 ----
                              n           2
                                         n
                3          3           2
               a          a         a b
         - 1/6 ---- + 1/2 ---- - 1/2 ----
                2          n          n
               n
I = limit(RS, n, inf); pretty(I)

                                      3        3
                                 1/3 b  - 1/3 a
%

echo off; diary off
```