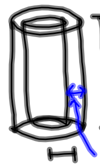
7.3 Cylindrical Shells



revolve around y-axis

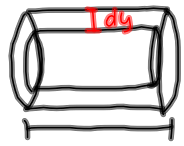
$h = f(x)$, dx, $r = x$

$V_{shell} = 2\pi r h\, dx$

$$\boxed{V_{shell} = 2\pi x f(x)\, dx}$$

• exam 1 is
Thurs Feb 16 [6.4–8.2]
• Review exam 1 is
Wed Feb 15
5:45–7:45 pm HELC 200

revolve around x-axis



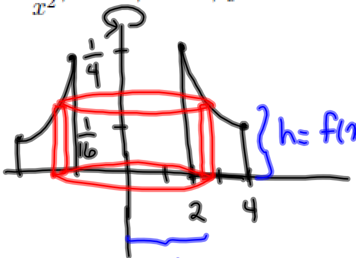
dy, $r = y$, $h = g(y)$

$V_{shell} = 2\pi r h\, dy$

$$\boxed{V_{shell} = 2\pi y\, g(y)\, dy}$$

1. Find the volume of the solid obtained by rotating the region bounded by the given curve(s) about the specified axis.

a.) $y = \frac{1}{x^2}$, $x = 2$, $x = 4$, $y = 0$ about the y axis.



$\frac{1}{4}$, $\frac{1}{16}$, $h = f(x)$, 2, 4, $r = x$

① washer with respect to y
split integral $\int_0^{\frac{1}{16}} + \int_{\frac{1}{16}}^{\frac{1}{4}}$

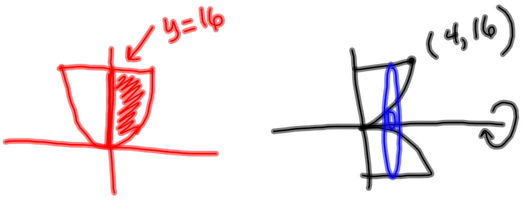
$$V = \int_2^4 2\pi \underbrace{(x)}_{r} \underbrace{\left(\frac{1}{x^2}\right)}_{h} dx = 2\pi \int_2^4 \frac{1}{x}\, dx$$

$$= 2\pi \ln x \Big|_2^4$$

$$= 2\pi \left[\ln 4 - \ln 2\right] = \boxed{2\pi \ln 2}$$

$\ln a - \ln b = \ln \frac{a}{b}$

b.) $y = x^2$, $y = 16$, $x = 0$ about the x-axis.



$y = 16$, $(4, 16)$

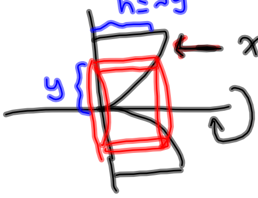
washers: $R = 16$, $r = x^2$

$$V = \int_0^4 \pi \left[R^2 - r^2\right] dx$$

$$= \int_0^4 \pi \left[(16)^2 - (x^2)^2\right] dx$$

$$= \_\_\_\_$$

Shells



$h = \sqrt{y}$, $x = \sqrt{y}$, y

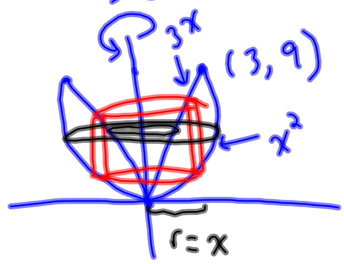
$$V = \int_0^{16} 2\pi y \sqrt{y}\, dy$$

$$= 2\pi \int_0^{16} y^{\frac{3}{2}}\, dy = 2\pi \cdot \frac{2}{5} y^{\frac{5}{2}} \Big|_0^{16} = \frac{4\pi}{5}\left(16^{\frac{5}{2}}\right)$$

$$= \frac{4\pi}{5}\left(4^5\right) = \boxed{\frac{4^6 \pi}{5}}$$

c.) $y = x^2$, $y = 3x$. Rotate around the y axis.

intersect at $(0,0)$ + $(3,9)$



$h = 3x - x^2$

$r = x$

washers

$V = \int_0^9 \pi\left[(\sqrt{y})^2 - \left(\frac{y}{3}\right)^2\right] dy$

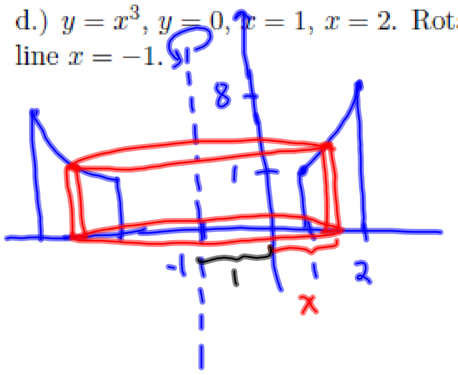
shells

$V = \int_0^3 2\pi x(3x - x^2)\,dx$

$= 2\pi \int_0^3 (3x^2 - x^3)\,dx = 2\pi\left(x^3 - \frac{x^4}{4}\right)\Big|_0^3$

$= \boxed{\frac{27\pi}{2}}$

d.) $y = x^3$, $y = 0$, $x = 1$, $x = 2$. Rotate around the line $x = -1$.



$r = 1 + x$

$h = f(x)$

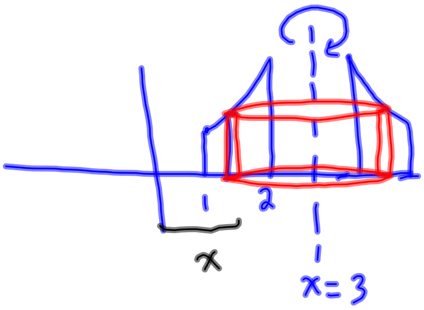
$V = \int_1^2 2\pi(x+1)(x^3)\,dx$

$= 2\pi \int_1^2 (x^4 + x^3)\,dx$

$= 2\pi\left(\frac{x^5}{5} + \frac{x^4}{4}\right)\Big|_1^2$

$= \boxed{\frac{199\pi}{10}}$

same region revolved around $x = 3$

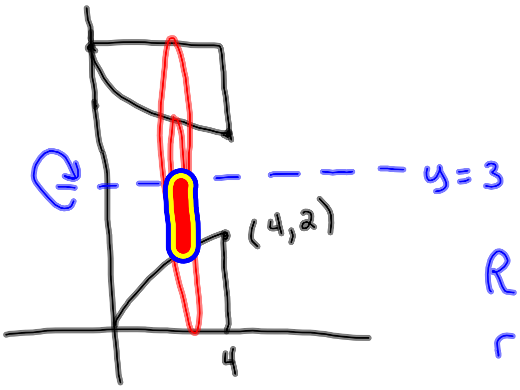


$r = 3 - x$ $\quad V = \int_1^2 2\pi(3-x)x^3\,dx$

$h = x^3$

$\vdots$

e.) $y = \sqrt{x}$, $x = 0$, $x = 4$, $y = 0$. Rotate around the line $y = 3$.



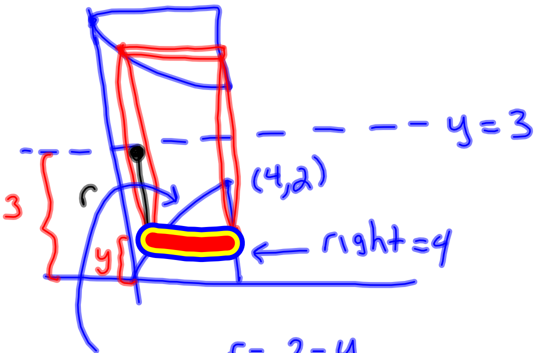
washers

$$V = \int_0^4 \pi\left[3^2 - (3-\sqrt{x})^2\right]dx$$

$$= \underline{\qquad}$$

$R = 3$

$r = 3 - \sqrt{x}$



left $= y^2$

$r = 3 - y$

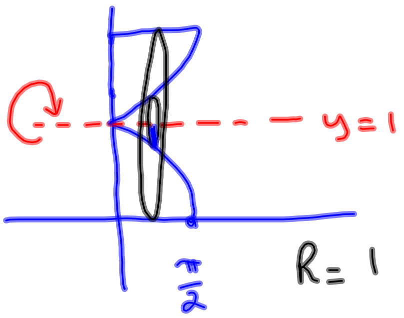
$h = 4 - y^2$

$$V = \int_0^2 2\pi(3-y)(4-y^2)\,dy$$

Foil & integrate

f.) $y = \cos x$, $y = 0$, $x = 0$, $x = \frac{\pi}{2}$. Rotate around the line $y = 1$. Now rotate around the line $x = \frac{\pi}{2}$. Do not evaluate either integral.

① rotate around $y=1$



$R = 1$

$r = 1 - \cos x$

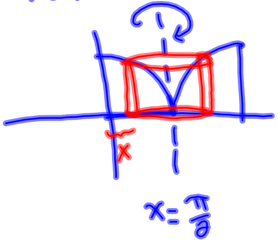
washers:

$$V = \int_0^{\frac{\pi}{2}} \pi\left[(1)^2 - (1-\cos x)^2\right]dx$$

② rotate around $x = \frac{\pi}{2}$

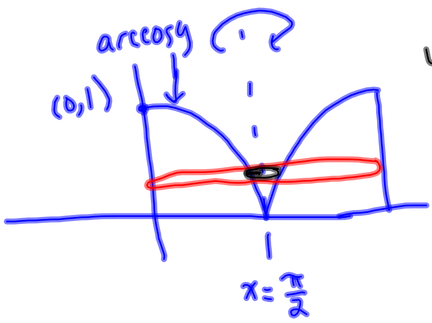
$r = \frac{\pi}{2} - x$

$h = \cos x$



shells

$$V = \int_0^{\frac{\pi}{2}} 2\pi\left(\frac{\pi}{2} - x\right)\cos x\,dx$$



washers

$R = \frac{\pi}{2}$, $r = \frac{\pi}{2} - \arccos y$

$$V = \int_0^1 \pi\left[\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{2} - \arccos y\right)^2\right]dy$$

2. How much work is done in lifting a 30 lb barbell from the floor to a height of 4 feet?

force

distance

If an object is moving a distance d feet or meters

under a constant force F, Newtons or pounds

Then the work done is $W = (F)(d)$

$$W = (30\text{ lbs})(4\text{ feet}) = \boxed{120\text{ ft-lbs}}$$

3. When a particle ~~is at a distance~~ x meters from the origin, a force of $\cancel{f(x) = 3x^2 + 2}$ Newtons acts on it. ~~How much work~~ is done in moving the object from ~~$x = 2$ to $x = 4$?~~

If the force is not constant, the work done in moving the object from $x=a$ to $x=b$ under a variable force is $W = \int_a^b f(x)\,dx$

$$W = \int_2^4 (3x^2+2)\,dx$$

$$= (x^3 + 2x)\Big|_2^4 = 64 + 8 - 8 - 4$$

$$= 60\text{ Nm or } 60\text{ J}$$

4. A spring has a natural length of 6 inches. If a 5-lb force is required to maintain it to a length of 18 inches, how much work is required to stretch it from 1 foot to 3 feet?

Hooke's law: The force required to hold a spring stretched x units beyond its natural length is

$$\boxed{f(x) = kx} \leftarrow \text{force function}$$

$\frac{1}{2}$ foot = natural length

holding 1 foot beyond natural length under a force of 5lbs

$$\underbrace{f(x)}_{5\text{lbs}} = k\underbrace{x}_{1\text{ foot}}$$

$$5 = k(1) \Rightarrow k = 5\ \frac{\text{lbs}}{\text{ft}}$$

$$\boxed{\text{variable force is } f(x) = 5x}$$

work to stretch 1 foot to 3 feet

subtract 1 & 3 from the natural length of $\frac{1}{2}$

$$W = \int_{\frac{1}{2}}^{\frac{5}{2}} 5x\,dx$$

$$W = \frac{5x^2}{2}\Bigg|_{\frac{1}{2}}^{\frac{5}{2}}$$

$$= \boxed{\frac{65}{4}\text{ ft lbs}}$$

5. Suppose 2 J of work is needed to stretch a spring 1 meter beyond its natural length. How much work is done in stretching this spring 3.5 m beyond it's natural length?

given: $W = 2$

$$\int_0^1 kx\,dx = 2 \Rightarrow k\frac{x^2}{2}\Big|_0^1 = 2$$

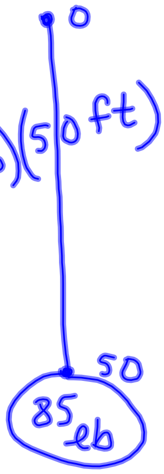
$$\boxed{k = 4}$$

force function is $f(x) = 4x$

$$W = \int_0^{3.5} 4x\,dx = 4\frac{x^2}{2}\Big|_0^{3.5} = 24.5\text{ J}$$

Rope pulling: A rope that weighs $k \frac{\text{lbs}}{\text{foot}}$ or $k \frac{N}{m}$ that is pulled t feet requires a work of $\int_0^t kx\,dx$

6. A heavy rope, 50 feet long, weighs 0.5 pounds per foot and hangs over the edge of a building 120 feet high. There is a 85 pound weight attached to the end of the rope. How much work is done in pulling the rope to the top of the building?



$F \cdot d = (85\,\text{lb})(50\,\text{ft}) = 4{,}250$ ft-lbs

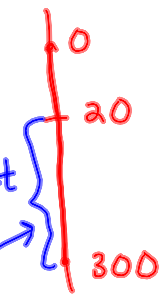
$$W_{total} = W_{rope} + W_{weight}$$

$$= \int_0^{50} 0.5\,x\,dx + 4250 \text{ ft lbs}$$

$$= \frac{1}{2}\frac{x^2}{2}\Big|_0^{50} + 4250$$

$$= \boxed{625 + 4250 \text{ ft lbs}} = \boxed{4875 \text{ ft-lbs}}$$

7. A 200 pound cable is 300 feet long and hangs vertically from the top of a tall building. How much work is required to pull 20 feet of the cable to the top of the building?



rope weighs $\frac{200 \text{ lbs}}{300 \text{ ft}} = \frac{2}{3} \frac{\text{lb}}{\text{ft}}$

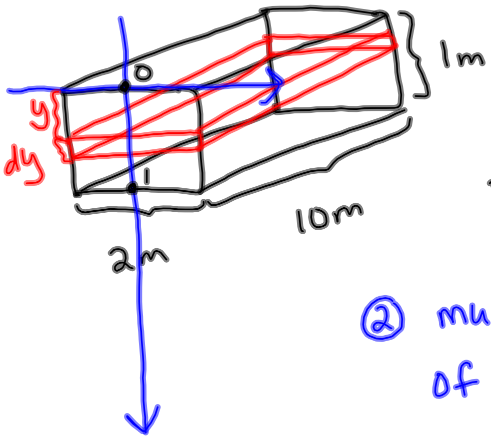
$$W = \int_0^{20} \frac{2}{3} x\,dx + \underbrace{(280)\left(\tfrac{2}{3}\right)(20)}_{\text{work to lift remaining rope}}$$

acts like a weight $(280\text{ft})\left(\frac{2}{3}\frac{\text{lb}}{\text{ft}}\right)$ moves 20 ft

$$= \frac{2}{3}\frac{x^2}{2}\Big|_0^{20} + \frac{11200}{3}$$

$$= \frac{1}{3}(20)^2 + \frac{11200}{3} = \boxed{\frac{11600}{3} \text{ ft-lbs}}$$

8. An aquarium 10 m long, 2 m wide and 1 m deep is full of water. Find the work required to pump half the water to the top of the aquarium.



① Find a formula for the volume of a slice of water.

$$V_s = (2m)(10m)(dy\,m) = 20dy\,m^3$$

② multiply by the weight density of water. $\rho g = 9800 \frac{N}{m^3}$

$$F_s = 20\rho g\,dy\,N$$

③ multiply by the distance the slice traveles to reach the top.

$d = y$ $\quad W_s = (20\rho g\,dy)(y)\,Nm$

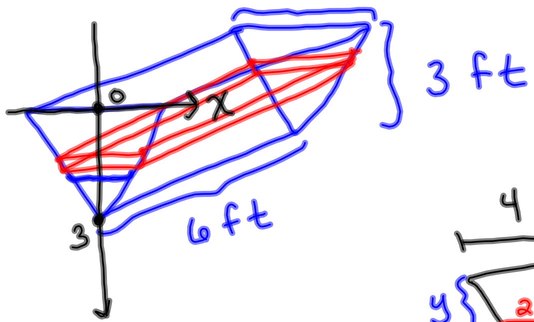
$$= 20\rho g\,y\,dy\,Nm$$

pump half water, $W = \int_0^{\frac{1}{2}} 20\rho g\,y\,dy$

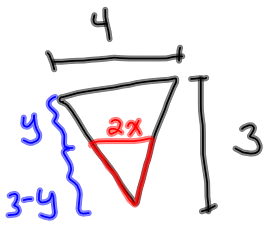
$$= 10\rho g\,y^2\Big|_0^{\frac{1}{2}} = \frac{10\rho g}{4}$$

$$= \boxed{\frac{5}{2}(9800)\,J}$$

9. A tank contains water and has the shape of a trough 6 feet long. The end of the trough is an isosceles triangle with height 3 feet and base length 4 feet. The vertex of the triangle is at the bottom. Find the work required to pump all of the water to the top of the tank.



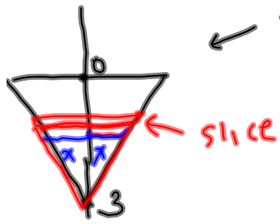
① $V_s = (2x)(6)(dy)\ ft^3$



$$\frac{2x}{3-y} = \frac{4}{3}$$

$$2x = \frac{4}{3}(3-y)$$

front of tank



$$\boxed{V_s = \frac{4}{3}(3-y)(6)dy = 8(3-y)dy}$$

② multiply by $\rho g = 62.5 \frac{lbs}{ft^3}$

$$F_s = 8\rho g(3-y)dy$$

③ multiply by distance $d = y$

$$W_s = 8\rho g(3-y)y\,dy$$

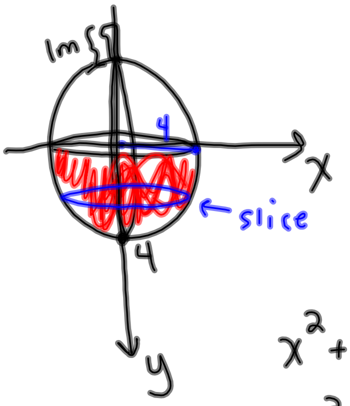
$$W = \int_0^3 8\rho g(3y-y^2)dy = 8\rho g\left(\frac{3y^2}{2} - \frac{y^3}{3}\right)\Big|_0^3$$

$$= 36\rho g \text{ ft-lbs}$$

$$= \boxed{(36)(62.5) \text{ ft-lbs}}$$

10. A tank in the shape of sphere with radius 4 m is half full of water. The water is pumped from a spout at the top of the tank that is 1 m high. Find the work done in pumping the water through the spout.

spout adds to distance



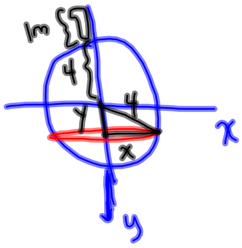
$V_s = \pi x^2 \, dy$

$x^2 + y^2 = 16$

$x^2 = 16 - y^2$

$V_s = \pi(16 - y^2)\, dy$

$F_s = \pi \rho g (16 - y^2)\, dy, \quad \rho g = 9800 \frac{N}{m^3}$



distance $d = y + 4 + 1 = y + 5$

$W_s = \pi \rho g (16 - y^2)(y+5)\, dy$

$W = \int_0^4 \pi \rho g (16 - y^2)(y+5)\, dy$