Numerical Integration: sec 8.8

1. Numerical Integration: Suppose I’d like to know \(\int_a^b f(x) \, dx\). There are three techniques of approximating an integral:

I. Midpoint Rule:
\[
\int_a^b f(x) \, dx \approx \Delta x [f(x_0) + f(x_1) + f(x_2) + \ldots + f(x_n)]
\]
where \(\Delta x = \frac{b-a}{n}\) and \(x_i\) is the midpoint of the \(i\)th subinterval.

II. Trapezoid Rule:
\[
\int_a^b f(x) \, dx \approx \frac{\Delta x}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + f(x_n)]
\]
where \(\Delta x = \frac{b-a}{n}\) and \(x_i\) are the points of the partition.

III. Simpson’s Rule:
\[
\int_a^b f(x) \, dx \approx \frac{\Delta x}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + \ldots + f(x_n)]
\]
where \(\Delta x = \frac{b-a}{n}\) and \(x_i\) are the points of the partition.

• Error Bound formulas: If you are asked to find an upper bound on the error, these formulas will be provided on the 152 common exam.

1. Error Bound for Midpoint Rule:
\[
|E_M| \leq \frac{K(b-a)^3}{24n^2}, \text{ where } K = \max|f''(x)| \text{ for } a \leq x \leq b
\]

2. Error Bound for Trapezoid Rule:
\[
|E_T| \leq \frac{K(b-a)^3}{12n^2}, \text{ where } K = \max|f''(x)| \text{ for } a \leq x \leq b
\]

3. Error Bound for Simpson’s Rule:
\[
|E_S| \leq \frac{K(b-a)^5}{180n^4}, \text{ where } K = \max|f^{(4)}(x)| \text{ for } a \leq x \leq b
\]

Improper Integrals: sec 8.9

2. Improper Integrals:

Case I: Integrals where one (or both) of the limits is infinite: Your goal is to determine whether the improper integral converges (finite value) or diverges (infinite value).

a.) \(\int_a^\infty f(x) \, dx = \lim_{t \to \infty} \int_a^t f(x) \, dx\)

b.) \(\int_{-\infty}^b f(x) \, dx = \lim_{t \to -\infty} \int_t^b f(x) \, dx\)

c.) \(\int_{-\infty}^\infty f(x) \, dx = \int_{-\infty}^a f(x) \, dx + \int_a^\infty f(x) \, dx\), then try to evaluate both integrals.

Case II: Integrals where there is a discontinuity on the interval \([a, b]\):

a.) Suppose \(f(x)\) is discontinuous at \(x = a\): Then
\[
\int_a^b f(x) \, dx = \lim_{t \to \infty} \int_t^b f(x) \, dx
\]

b.) Suppose \(f(x)\) is discontinuous at \(x = b\): Then
\[
\int_a^b f(x) \, dx = \lim_{t \to -\infty} \int_a^t f(x) \, dx
\]

c.) If \(f(x)\) is discontinuous at some \(c\) where \(a < c < b\), then
\[
\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx\), then try to evaluate both integrals.

• Comparison Theorem for Improper Integrals:

a.) Suppose \(f(x)\) and \(g(x)\) are continuous, positive functions on the interval \([a, \infty)\). Also, suppose that \(f(x) \geq g(x)\) on the interval \([a, \infty)\). Then:

(i) If \(\int_a^\infty f(x) \, dx\) converges, so does \(\int_a^\infty g(x) \, dx\).

(Note: If \(\int_a^\infty f(x) \, dx\) diverges, no conclusion can be drawn about \(\int_a^\infty g(x) \, dx\).)

(ii) If \(\int_a^\infty g(x) \, dx\) diverges, so does \(\int_a^\infty f(x) \, dx\).

(Note: If \(\int_a^\infty g(x) \, dx\) converges, no conclusion can be drawn about \(\int_a^\infty f(x) \, dx\).)

Note: The way you choose the comparison function: You take the largest part of the numerator over the largest part of the denominator on the interval \([a, \infty)\). Once you find the comparison function, you must determine the direction of the inequality, then integrate the comparison function and draw the correct conclusion.
Differential Equations: sec 9.1

3. Def: A differential equation is an equation that contains an unknown function and some of its derivatives. Your primary goal is to try to solve the differential equation.

- A differential equation is separable if it is in the form \( Q(y)dy = P(x)dx \). To solve such an equation, integrate both sides.

  \[
  \frac{dy}{dx} = \frac{4x^2}{2y^5} \quad \text{separate it:} \quad 2y^5dy = 4x^2dx, \therefore \quad \frac{2}{3}y^3 = \frac{4}{3}x^3 + C \quad \text{Then solve for} \quad y. \quad \text{You may have an initial condition which allows you to solve for} \quad C: \quad \text{Suppose} \quad y(2) = 3, \quad \text{then} \quad \frac{2}{3}3^3 = \frac{4}{3}2^3 + C, \quad \text{solve for} \quad C, \quad \text{then solve for} \quad y.
  \]

- Brine Problems: Suppose a tank contains \( L \) Liters of salt water (could contain no salt at time \( t = 0 \)). Now let’s suppose a salt concentration \( I \) is going into the tank at a given rate \( R \), the solution is continually stirred and it is exiting the tank at the same rate. Then if \( y = y(t) \) is the amount of salt in the tank at time \( t \), then \( \frac{dy}{dt} = (I)*R - \frac{Y}{L}*R \), and \( y(0) = \) amnt of salt in the tank at time \( t = 0 \). Then solve for \( y \).

Differential Equations: sec 9.2

4. Linear differential equations

- A differential equation is linear if it is in the form \( \frac{dy}{dx} + P(x)y = Q(x) \). It is important that you recognize which variable is independant and which is dependant. If your equation contains \( \frac{dy}{dx} \), then the independant variable is \( x \); the dependant variable is \( y \).

- To solve a linear differential equation, you must first find the integrating factor \( I(x) = e^{\int P(x)dx} \).

- Next, multiply both sides of the differential equation by \( I(x) \): \( I(x)\left(\frac{dy}{dx} + P(x)y\right) = I(x)Q(x) \), which then becomes \( (yI(x))' = I(x)Q(x) \). Next, integrate both sides and then solve for \( y \).

Differential Equations: sec 9.3

5. There are three possible formulas which gives the length of a curve:

a.) If \( y = f(x), a \leq x \leq b \), then the length of the curve from \( x = a \) to \( x = b \) is \( L = \int_{a}^{b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \) \( dx \)

b.) If \( x = g(y), c \leq y \leq d \), then the length of the curve from \( y = c \) to \( y = d \) is \( L = \int_{c}^{d} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \) \( dy \)

c.) If \( x = f(t) \) and \( y = g(t), \alpha \leq t \leq \beta \), then the length of the curve from \( t = \alpha \) to \( t = \beta \) is

\[
L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \) \( dt \)

Surface Area of Revolution: sec 9.4

6. Revolution around the \( x \) axis:

a.) If the curve \( y = f(x), a \leq x \leq b \) is revolved around the \( x \) axis, then the resulting surface area is given by

\[
SA = 2\pi \int_{a}^{b} f(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \) \( dx \)

b.) If the curve \( x = g(y), c \leq y \leq d \) is revolved around the \( x \) axis, then the resulting surface area is given by

\[
SA = 2\pi \int_{c}^{d} y \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \) \( dy \)

c.) If the curve \( x = f(t) \) and \( y = g(t), \alpha \leq t \leq \beta \), is revolved around the \( x \) axis, then the resulting surface area is given by

\[
SA = 2\pi \int_{\alpha}^{\beta} g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \) \( dt \)

7. Revolution around the \( y \) axis:

a.) If the curve \( y = f(x), a \leq x \leq b \) is revolved around the \( y \) axis, then the resulting surface area is given by

\[
SA = 2\pi \int_{a}^{b} x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \) \( dx \)

b.) If the curve \( x = g(y), c \leq y \leq d \) is revolved around the \( y \) axis, then the resulting surface area is given by

\[
SA = 2\pi \int_{c}^{d} \frac{g(y)}{\sqrt{1 + \left(\frac{dx}{dy}\right)^2}} \) \( dy \)

c.) If the curve \( x = f(t) \) and \( y = g(t), \alpha \leq t \leq \beta \), is revolved around the \( y \) axis, then the resulting surface area is given by

\[
SA = 2\pi \int_{\alpha}^{\beta} \frac{f(t)}{\sqrt{1 + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}} \) \( dt \)
Moments and Centers of Gravity: sec 9.5

8. If we have a system of \( n \) particles with masses \( m_1, m_2, ..., m_n \) located at the points \( x_1, x_2, ..., x_n \) on the \( x \) axis, then

\[
\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}
\]

9. If we have a system of \( n \) particles with masses \( m_1, m_2, ..., m_n \) located at the points \((x_1, y_1), (x_2, y_2), ..., (x_n, y_n)\) in the \( x-y \) plane.

- The **moment of the system about the \( y \) axis** is
  \[ M_y = \sum_{i=1}^{n} m_i x_i \]
  This measures the tendency of the system to rotate about the \( y \) axis.

- The **moment of the system about the \( x \) axis** is
  \[ M_x = \sum_{i=1}^{n} m_i y_i \]
  This measures the tendency of the system to rotate about the \( x \) axis.

- The **center of mass** is \((\bar{x}, \bar{y})\) where
  \[
  \bar{x} = \frac{M_y}{\sum_{i=1}^{n} m_i} \quad \text{and} \quad \bar{y} = \frac{M_x}{\sum_{i=1}^{n} m_i}
  \]

10. Now we have a function \( y = f(x) \) with uniform density \( \rho \) on the interval \([a, b]\).

- The moment about the \( y \)-axis is:
  \[ M_y = \rho \int_{a}^{b} x f(x) \, dx \]

- The moment about the \( x \)-axis is:
  \[ M_x = \rho \int_{a}^{b} \frac{1}{2} (f(x))^2 \, dx \]

- The \( x \) coordinate of the centroid is
  \[ \bar{x} = \frac{1}{A} \int_{a}^{b} x f(x) \, dx \], where \( A \) is the area of the region.

- The \( y \) coordinate of the centroid is
  \[ \bar{y} = \frac{1}{A} \int_{a}^{b} \frac{1}{2} [f(x)]^2 \, dx \], where \( A \) is the area of the region.

Hydrostatic Pressure and Force: sec 9.6

12. General formula: The hydrostatic force on a horizontal plate is
  \[ F = \rho g d A \]
  where \( \rho g \) is the weight density of the liquid, \( d \) is the depth of the plate, and \( A \) is the area of the plate. If you are finding the force on a vertical plate, finding the hydrostatic force requires integration.
  Take a small vertical strip of water with area \( A \). Then
  \[ F = \int \rho g d A \]
  where \( d \) is the depth of the vertical strip.
  The limits of integration depends on how you define your axes.