Homework 11

Math 171H (section 201), Fall 2023

This homework is due on **Tuesday, October 31** at the start of class. (Turn in answers to questions 1–8.)

0. *(This problem is not to be turned in.)*
   (a) Read Sections 4.2–4.3
   (b) Find all critical numbers of the following functions:
   - $f(x) = x^3 + 6x^2 + 3x - 1$
   - $g(x) = x + \sin x$
   - $h(x) = \frac{1}{(1-x)^2}$

   (c) Complete the following claims, and then give a proof:
   - $f(x)$ is **increasing** on an interval I if and only if $-f(x)$ is __________ on I.
   - $f(x)$ is **concave up** on an interval I if and only if $-f(x)$ is __________ on I.

1. Determine the value(s) of m and b that make the following function **differentiable**:

$$f(x) = \begin{cases} \arctan x & \text{if } x < 1 \\ mx + b & \text{if } x \geq 1 \end{cases}$$

2. (a) Does $f(x) = 1/x$ satisfy the hypotheses of the Mean Value Theorem on the interval $[1, 4]$? If yes, then find all values c that satisfy the conclusion of the theorem. If not, explain why not.
   (b) Does $f(x) = |x - 1|$ satisfy the hypotheses of the Mean Value Theorem on the interval $[0, 2]$? If yes, then find all values c that satisfy the conclusion of the theorem. If not, explain why not.
   (c) Does $f(x) = |x - 1|$ satisfy the hypotheses of the Mean Value Theorem on the interval $[1, 4]$? If yes, then find all values c that satisfy the conclusion of the theorem. If not, explain why not.

3. Assume that $f'(x) = (x - 1)(x - 2)$.
   (a) On what intervals is $f(x)$ increasing?
   (b) List all x-values at which $f(x)$ has a **local maximum**.
   (c) List all x-values at which $f(x)$ has a **local minimum**.
   (d) Can you determine where $f(x) > 0$ and where $f(x) < 0$ and where $f(x) = 0$? Explain.

4. Prove or disprove:
   (a) If $f(x)$ is differentiable and decreasing on an open interval (a, b), then $f'(x) < 0$ on (a, b).
   (b) If $f'(a) < 0$ (for some real number a), then $f(x)$ is decreasing on an interval containing a.

5. Assume that $f(x)$ is decreasing on (a, b).

   (a) Give an example of such a function (and an interval (a, b)), for which $f(x)$ is **continuous**.

   (b) Give an example of such a function (and an interval (a, b)), for which $f(x)$ is **discontinuous**.

   (c) Prove that $f(x)$ (with domain (a, b)) has an inverse.

   (d) Is the inverse $f^{-1}(x)$ decreasing or increasing or neither? Prove your answer.

6. For each of the following, give an example of a function (a sketch of the graph is fine) with the listed properties:

   (a) $f(x)$ is increasing and concave up (on all of the domain)

   (b) $f(x)$ is increasing and concave down (on all of the domain)

   (c) $f(x)$ is decreasing and concave up (on all of the domain)

   (d) $f(x)$ is decreasing and concave down (on all of the domain)

7. A **fixed point** of $f(x)$ is a value c at which $f(c) = c$. Prove the following: *If $f(x)$ is differentiable on an interval and has at least 2 fixed points (in that interval), then $f'(a) = 1$ for some a in the interval.*

8. Assume that $f''(x) > 0$ on an interval (a, b). The goal is this problem is to prove that $f(x)$ is **concave up** on (a, b).

   (a) Prove that $f'(x)$ is increasing on (a, b).

   (b) Show that $f(x)$ is concave up on (a, b) if and only if $f(x) > f(c) + f'(c)(x - c)$ for all c in (a, b) and all x in $(a, b) \setminus \{c\}$. (HINT: $f(c) + f'(c)(x - c)$ is the equation of a tangent line.)

   (c) Let c be in (a, b). Prove that if there exists d in $(a, b) \setminus \{c\}$ such that $f(d) \le f(c) + f'(c)(d - c)$, then $f'(x)$ is not increasing on (a, b). (HINT: Apply the Mean Value Theorem to $[c, d]$ if $c < d$ or to $[d, c]$ if $d < c$.)

   (d) Use (a)–(c) to conclude that $f(x)$ is concave up on (a, b).