Homework 11
Math 415 (section 502), Fall 2015

This homework is due on Thursday, November 12. You may cite results from class, as appropriate.

0. (This problem is not to be turned in.)
   (a) Read Section 18.
   (b) Section 18 # 38, 44, 55

1. True/false (No proofs necessary for this problem.)
   (a) $S_3$ is a simple group.
   (b) $A_3$ is a simple group.
   (c) $A_5$ is a simple group.
   (d) If $f : X \to Y$ is any function, and $U \subset X$, then $f^{-1}[f[X]] = X$.
   (e) If $f : X \to Y$ is any function, and $V \subset Y$, then $f[f^{-1}[V]] = V$.
   (f) If $f : X \to Y$ is any function, and $V \subset Y$, and $f$ is onto (surjective), then $f[f^{-1}[V]] = V$.

2. Let $R$ be a ring with unity, and let $U$ denote the set of all units in $R$. Prove that $U$ is closed under the operation of multiplication, and furthermore that it is a group (under multiplication).

3. Describe all ring homomorphisms $\mathbb{Z} \to \mathbb{Z}$.

4. Up to isomorphism, are there any finite simple abelian groups besides $\mathbb{Z}_1$ and $\mathbb{Z}_p$ (where $p$ is a prime number)? Give a proof.

5. The aim of this problem is to prove the following: if $M$ is a proper normal subgroup of $G$, and $G/M$ is simple, then $M$ is a maximal normal subgroup of $G$. Accordingly, assume that $M$ is a proper normal subgroup of $G$, and $G/M$ is simple.
   (a) Let $\gamma : G \to G/M$ be the usual homomorphism given by $x \mapsto xM$. Prove that if a normal subgroup $N$ of $G$ satisfies $M \subseteq N \subseteq G$, then $\gamma[N] \subseteq G/M$. (Hint: use a theorem that tells us about the image of a normal subgroup.)
   (b) Use (a) to prove that either $\{M\} = \gamma[N]$ or $\gamma[N] = G/M$. (Hint: $G/M$ is simple.)
   (c) Use (b) to prove that either $M = N$ or $N = G$. (Hint: the containments $\subset$ are by assumption; for $\supset$, use (b), but the true/false problems are supposed to warn you that you can not state that $\gamma^{-1}[\gamma[N]] = N$ without proof).

6. (a) Draw the lattice of the following subgroups of $\mathbb{Z}$: $\{0\}, 5\mathbb{Z}, 8\mathbb{Z}, 9\mathbb{Z}$, and all subgroups that contain 12. Highlight or circle the subgroups that contain 12.
   (b) Draw the lattice of all subgroups of $\mathbb{Z}/12\mathbb{Z}$.
   (c) Use (a) and (b) to state a conjecture: for a group $G$ with a normal subgroup $N$, then $K \mapsto K/N$ defines a bijection between the set of subgroups of $G$ that contain $N$ and the set of ____________.
   (d) (Challenge problem – optional!) Prove the conjecture you stated in (c).

7. Section 18 # 6, 12, 27, 33, 41