Homework 8
Math 415 (section 502), Fall 2015

This homework is due on Thursday, October 22. You may cite results from class/homework/exam.

0. *(This problem is not to be turned in.)*
   (a) Read Section 13.
   (b) Section 13 \# 1–28, 45 (do as many as you can; some are to be turned in – see below)
   (c) Prove that if \( H \) is a subgroup of a group \( G \), then the inclusion function \( i : H \to G \) given by \( i(x) := x \) is a homomorphism. What is the kernel?
   (d) List all homomorphisms \( \mathbb{Z}_{10} \to A_4 \), and their kernels and images.
   (e) List all homomorphisms \( \mathbb{Z} \to A_4 \) for which the kernel is \( \langle 2 \rangle \).
   (f) If \( G, H, \) and \( K \) are finite groups of order \( m, n, \) and \( p \), respectively, then what is the order of \( G \times H \times K \)?
   (g) If \( G \) is an infinite group, and \( H \) is a finite group, does it follow that \( G \times H \) is an infinite group?
   (h) Prove or disprove: *if \( \phi : G \to G' \) is a homomorphism, and \( G' \) is abelian, then \( G \) is abelian.*
   (i) Prove or disprove: *if \( \phi : G \to G' \) is a homomorphism, and \( G \) is abelian, then \( G' \) is abelian.*
   (j) Prove or disprove: *if \( \phi : G \to G' \) is a surjective homomorphism, and \( G' \) is abelian, then the kernel of \( \phi \) is abelian.*
   (k) Prove or disprove: *if \( \phi : G \to G' \) is a homomorphism, and \( G' \) is cyclic, then \( G \) is cyclic.*

1. Let \( G \) be a group, and let \( g \in G \). Consider the function \( \phi : G \to G \) given by \( \phi(x) = gxg^{-1} \).
   (a) Prove that \( \phi \) is a homomorphism.
   (b) Determine the kernel of \( \phi \).
   (c) Is \( \phi \) an automorphism? Give a proof. (Recall that an *automorphism* of a group \( K \) is an isomorphism from \( K \) to \( K \).)

2. Let \( G \) and \( K \) be groups. Let \( \pi : G \times K \to G \) be the projection function \( \pi(g, k) := g \).
   (a) Prove that \( \pi \) is a homomorphism.
   (b) Prove that kernel of \( \pi \) is isomorphic to \( K \).
3. (a) Prove or disprove: if $\phi : G \to G'$ is a homomorphism, and $G'$ is infinite, then $G$ is infinite.

(b) Prove or disprove: if $\phi : G \to G'$ is a homomorphism, $G$ is infinite, and $G'$ is finite, then $\ker(\phi)$ is infinite.

4. (a) Explain how the symmetric group $S_n$ can be viewed as a subgroup of $S_m$ for any $m \geq n$.

(b) Are (17) and (1237) in the same left coset of (the subgroup) $S_6$ in the group $S_7$? Explain. (Hint: Recall the criterion for when 2 cosets are equal.)

(c) Are (27) and (1237) in the same left coset of (the subgroup) $S_6$ in the group $S_7$? Explain.

5. Section 13 # 10, 22, 29, 32, 40, 50, 52

6. Section 14 # 6

7. (Challenge problem – optional!)

(a) Prove or disprove: if $G$ is an abelian group that is not cyclic, then $G$ contains a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ for some prime number $p$. (Last week’s homework did the case when $G$ is finite.)

(b) Prove or disprove: for a subgroup $H$ of a group $G$, every left coset of $H$ contains the identity element of $G$.

(c) Prove or disprove: for a subgroup $H$ of a group $G$, if two left cosets of $H$ intersect, then they are equal.

(d) Section 13 # 53