Homework 9
Math 415 (section 502), Fall 2015

This homework is due on Thursday, October 29. You may cite results from class/homework/quiz/exam.

0. (This problem is not to be turned in.)
   (a) Read Section 14.
   (b) Section 14 # 6, 22, 35, 38, 40

1. True/False. (No proofs necessary for this problem.)
   (a) $\mathbb{Z}_3 \times \mathbb{Z}_{10}$ is cyclic.
   (b) $\mathbb{Z} \times \mathbb{Z}_5$ is cyclic.
   (c) $\mathbb{Z}_6 \times \mathbb{Z}_{21}$ is cyclic.
   (d) $\mathbb{Z}_6 \times \mathbb{Z}_2$ is isomorphic to $\mathbb{Z}_{12}$.
   (e) Every subgroup of $\mathbb{Z} \times \mathbb{Z}_5 \times \mathbb{R}$ is normal.
   (f) If $\phi : G \to G'$ is a surjective homomorphism, then $G/\ker(\phi) \cong G'$.
   (g) Assume $H \trianglelefteq G$ (i.e., $H$ is a normal subgroup of $G$). If $H$ is abelian and $G/H$ is abelian, then $G$ is abelian.
   (h) Assume $H \trianglelefteq G$. If $H$ is finite and $G/H$ is finite, then $G$ is finite.
   (i) For any group $G$, the factor group $G/G$ is isomorphic to $\mathbb{Z}_1$.
   (j) For any group $G$, the factor group $G/\{e\}$ is isomorphic to $G$.

2. (No proofs necessary for this problem.)
   (a) What is the order of $(1, 1)$ in $\mathbb{Z}_{10} \times \mathbb{Z}_{15}$?
   (b) Why is $\langle (1, 1) \rangle$ a normal subgroup of $\mathbb{Z}_{10} \times \mathbb{Z}_{15}$?
   (c) Explain briefly why $\mathbb{Z}_{10} \times \mathbb{Z}_{15}/\langle (1, 1) \rangle$ is isomorphic to a direct product of cyclic groups of prime-power order.
   (d) What is the order of $\mathbb{Z}_{10} \times \mathbb{Z}_{15}/\langle (1, 1) \rangle$?
   (e) Write down an isomorphism between $\mathbb{Z}_{10} \times \mathbb{Z}_{15}/\langle (1, 1) \rangle$ and a product as described in part (b). Show your work.

3. Assume $H \trianglelefteq G$. Prove that if $G$ is finite, then $G/H$ is finite. (Hint: Prove that $|G/H| = \frac{|G|}{|H|}$.)

4. Let $G$ be the group of all functions $\mathbb{R} \to \mathbb{R}$. In class, we saw that $H := \{ f \in G \mid f(5) = 0 \}$ is a normal subgroup of $G$. Use the fundamental homomorphism theorem to prove that $G/H \cong \mathbb{R}$. 
5. Let $G$ be a group. Recall that $\text{Aut}(G)$, the set of all automorphisms of $G$, forms a group under composition. Recall from a previous homework that for all $g \in G$, the function $i_g : G \to G$ given by $i_g(x) := gxg^{-1}$ is an automorphism of $G$. Let $I_G := \{i_g \mid g \in G\}$.

(a) Prove that $I_G \leq \text{Aut}(G)$.
(b) Prove that $I_G \unlhd \text{Aut}(G)$.

6. Assume that $H$ and $K$ are both normal subgroups of a group $G$ and that $K \subset H$. Prove that $K$ is a subgroup of $H$ and, further, that $K$ is a normal subgroup of $H$.

7. Section 14 # 14, 23

8. (Challenge problem – optional!)

(a) Section 14 #39
(b) Let $H$ be a subgroup of $G$. Prove that if the number of left cosets of $H$ in $G$ is 2, then $H \unlhd G$
(c) Let $G$ be the group of permutations of $\mathbb{Z}$. Let

$$H := \{\phi \in G \mid \phi(a) = a \text{ for all } a \leq 0\} .$$

Let $\sigma \in G$ be the permutation defined by $\sigma(a) := a + 1$. Prove the following:
(1) $H \leq G$,
(2) $\sigma h \sigma^{-1} \in H$ for all $h \in H$, and then conclude from the exam that $\sigma H \subseteq H \sigma$,
(3) $\sigma H \subsetneq H \sigma$, and then conclude that $H$ is not a normal subgroup of $G$. 