Divergence Test states if

$\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ may or may not converge.

How to determine what test to apply to determine whether $\sum_{n=1}^{\infty} a_n$ converges:

① Does $\lim_{n\to\infty} a_n = 0$? If not, series diverges

If so:

① Is $\{a_n\}$ (eventually) positive?

If so:

① integral test only use if $\{a_n\}$ are decreasing and easily integratable.

② comparison test

③ limit comparison test

Section 10.3

1. Determine whether the following series converge or diverge.

a.) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

① T.D. $\lim_{n\to\infty} \frac{1}{n\ln n} = 0 \rightarrow$ Test fails.

② $\frac{1}{n\ln n} > 0$, decreasing & integratable

use integral test $\int_2^{\infty} \frac{dx}{x\ln x}$

$u = \ln x$

$du = \frac{dx}{x}$

$\int \frac{du}{u} = \ln|u|$

$= \ln|\ln x|$

$= \ln|\ln x| \Big|_2^{\infty}$

$= \ln|\ln \infty| - \ln|\ln 2|$

$= \infty \rightarrow$ integral diverges, so does $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$

b.) $\sum_{n=2}^{\infty} n^2 e^{-n^3}$ $= \sum_{n=2}^{\infty} \frac{n^2}{e^{n^3}}$

① $\lim_{n\to\infty} \frac{n^2}{e^{n^3}} = 0 \rightarrow$ T.D. fails

② $\frac{n^2}{e^{n^3}} > 0$ and decreasing

easy to integrate

$\int_2^{\infty} x^2 e^{-x^3} dx$

$u = -x^3$

$du = -3x^2 dx$

$-\frac{1}{3}\int e^u du = -\frac{1}{3}e^u$

$= -\frac{1}{3}e^{-x^3}$

$= -\frac{1}{3}e^{-x^3} \Big|_2^{\infty}$

$= \cancel{-\frac{1}{3}e^{-\infty}}^{\,0} + \frac{1}{3}e^{-8}$

$= \frac{1}{3e^8} < \infty$

$\boxed{\text{integral converges, so does } \sum_{n=2}^{\infty} n^2 e^{-n^3}}$

p-series test: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p>1$, diverges if $p \le 1$

$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ divergent p-series $p=\frac{1}{2}<1$

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ convergent p-series $p=3>1$

comparison Test (CT) If $0 \le a_n \le b_n$ (eventually)

① if $\sum_{n=1}^{\infty} b_n$ converges $\rightarrow \sum_{n=1}^{\infty} a_n$ also converges.

② if $\sum_{n=1}^{\infty} a_n$ diverges $\rightarrow \sum_{n=1}^{\infty} b_n$ also diverges.

If larger series diverges, test fails

If smaller series converges, Test fails

c.) $\sum_{n=1}^{\infty} \frac{n^4}{n^8+n^2+1} \le \sum_{n=1}^{\infty} \frac{n^4}{n^8} = \sum_{n=1}^{\infty} \frac{1}{n^4}$ ← converges by p-series $p=4>1$

not easy to integrate, but is positive so try C.T.

larger series converges, so does smaller by C.T.

d.) $\sum_{n=5}^{\infty} \frac{1}{n-2\sqrt{n}}$ $\quad \frac{1}{n-2\sqrt{n}} > 0$ not integratable, try C.T.

$$\sum_{n=5}^{\infty} \frac{1}{n-2\sqrt{n}} \ge \sum_{n=5}^{\infty} \frac{1}{n}$$ ← divergent p-series $p=1$

smaller diverges, so does larger by CT

e.) $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n\sqrt{n}}$ T.D. fails bc $\lim_{n\to\infty} \frac{\sin^2 n}{n\sqrt{n}} = 0$

$\frac{\sin^2 n}{n\sqrt{n}} > 0$, not easy to integrate
try CT

$$\sum_{n=1}^{\infty} \frac{\sin^2 n}{n\sqrt{n}} \le \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$$ ← convergent p-series $p = \frac{3}{2} > 0$

larger converges so does smaller.

f.) $\sum_{n=1}^{\infty} \frac{n^2 - n}{n^3 + 7n}$ T.D. fails

series of positive terms
not easy to integrate

C.T. $$\sum_{n=1}^{\infty} \frac{n^2-n}{n^3+7n} \le \sum_{n=1}^{\infty} \frac{n^2}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n}$$

divergent p-series.

larger diverges, CT fails

LCT (limit comparison test)

$0 \le a_n$ + $0 \le b_n$.

and $$\boxed{\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0 \ne \infty}$$

then either both converge or both diverge.

f.) $\sum_{n=1}^{\infty} \frac{n^2 - n}{n^3 + 7n}$ Try LCT with $\sum_{n=1}^{\infty} \frac{1}{n}$

a_n b_n

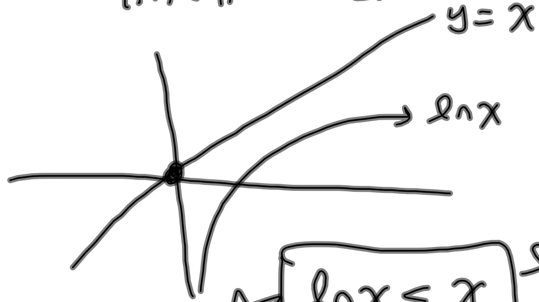
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{\frac{n^2-n}{n^3+7n}}{\frac{1}{n}} \right) = \lim_{n\to\infty} \frac{n^3 - n^2}{n^3 + 7n}$$

$$= 1 > 0 \ne \infty$$

Both series $\boxed{\text{diverges}}$ because $\sum_{n=1}^{\infty} \frac{1}{n}$ $\boxed{\text{div}}$

g.) $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

T.D. fails, positive, not easily integratable.



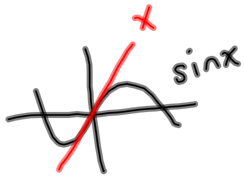
$$\sum_{n=2}^{\infty} \frac{1}{\ln n} \geq \sum_{n=2}^{\infty} \frac{1}{n}$$

$\boxed{\ln x \leq x}$

smaller diverges by p-series, $p=1$

larger also diverges

h.) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$



$\sin x \approx x$ if x near 0

$\sin\left(\frac{1}{n^2}\right) \approx \frac{1}{n^2}$ for large n.

$\cancel{\sum \sin(n)}$

$\sin\left(\frac{1}{n^2}\right) \geq 0$

T.D. fails

$\lim_{n\to\infty} \sin\left(\frac{1}{n^2}\right) = \sin 0 = 0$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin \frac{1}{n^2}}{\frac{1}{n^2}}$$

LCT with $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$= 1 > 0 \neq 0$

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$$

Both series $\boxed{\text{converges}}$

Because $\sum_{n=1}^{\infty} \frac{1}{n^2}$ $\boxed{\text{conv}}$

2. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

a.) Find the sum of the first 5 terms. $\rightarrow S_5$

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by p-series $\rightarrow$ sum exists.

$S_5 = a_1 + a_2 + a_3 + a_4 + a_5$

$$\boxed{S_5 = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3}}$$

b.) Estimate the error in using the sum of the first 5 terms to approximate the sum of the series.

Remainder estimate for the integral test:

If $\sum_{n=1}^{\infty} a_n$ was shown to be convergent by the integral test (this includes p-series and comparison test)

Then $R_n = S - S_n \le \int_n^{\infty} f(x)\,dx$, where $f(n) = a_n$.

$$R_5 = S - S_5 \le \int_5^{\infty} \frac{dx}{x^3} = -\frac{1}{2x^2}\Big|_5^{\infty}$$

$$= \cancelto{0}{-\frac{1}{2(\infty)^2}} + \frac{1}{2(5)^2}$$

$$\boxed{\therefore R_5 \le \frac{1}{50}}$$

c.) Find the sum correct to 10 decimal places.

① Find n so that $R_n \le 10^{-10}$

$$\underbrace{R_n \le \int_n^{\infty} \frac{dx}{x^3} \overset{\text{find } n}{\le} \frac{1}{10^{10}}}$$

$$-\frac{1}{2x^2}\Big|_n^{\infty} \le \frac{1}{10^{10}}$$

$$\frac{1}{2n^2} \le \frac{1}{10^{10}}$$

$$\frac{10^{10}}{2} \le n^2$$

$$\sqrt{\frac{10^{10}}{2}} \le n$$

$$70710.6 \le n$$

$\boxed{n \text{ must be at least } 70711}$

② use S_{70711} to approximate $\sum_{n=1}^{\infty} \frac{1}{n^3}$

$$\boxed{S_{70711} = 1 + \frac{1}{2^3} + \dots + \frac{1}{(70711)^3}}$$

3. Consider $\sum_{n=1}^{\infty} \frac{3+\cos n}{n^5}$

a.) Prove the series converges.

$\frac{3+\cos n}{n^5} > 0$ Try C.T.

$$\sum_{n=1}^{\infty} \frac{3+\cos n}{n^5} \leq \sum_{n=1}^{\infty} \frac{4}{n^5}$$

convergent p-series $p=5>1$

larger converges so does smaller.

b.) Approximate the sum of the series using s_6.

$$S_6 = a_1 + a_2 + \cdots + a_6$$

$$= \frac{3+\cos(1)}{1^5} + \cdots + \frac{3+\cos(6)}{6^5}$$

c.) Estimate the error in using s_6 to approximate the sum of the series.

since $\sum_{n=1}^{\infty} \frac{3+\cos n}{n^5}$ was shown to be convergent by a comparison test,

$$R_n \leq \int_n^{\infty} f(x)\,dx$$

$$R_6 \leq \int_6^{\infty} \frac{3+\cos x}{x^5}\,dx \leq \int_6^{\infty} \frac{4}{x^5}\,dx$$

$$= -\frac{1}{x^4}\Big|_6^{\infty}$$

$$= \frac{1}{6^4}$$

$$\boxed{\therefore\ R_6 \leq \frac{1}{6^4}}$$

Alternating series Test (AST)

If $\sum_{n=1}^{\infty} (-1)^n a_n$ satisfies: ① $a_{n+1} \le a_n$

then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges. ② $\lim_{n\to\infty} a_n = 0$

Section 10.4

4. Use the alternating series test to determine whether $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ converges.

① T.D. $\lim_{n\to\infty} \frac{(-1)^n}{\sqrt{n+1}} = 0$

AST: show $\left\{ \frac{1}{\sqrt{n+1}} \right\}$

① $a_{n+1} \le a_n \to \frac{1}{\sqrt{n+2}} \le \frac{1}{\sqrt{n+1}}$ ✓

② $\lim_{n\to\infty} a_n = 0 \to \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$ ✓

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ converges.

5. Determine whether the following series converge absolutely, converge conditionally, or diverge.

a.) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\sqrt{n}}$

① If $\sum_{n=1}^{\infty} |a_n|$ converges, Then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

② If $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ diverges, then $\sum_{n=1}^{\infty} a_n$ is conditionally convergent.

use AST to establish convergence:

show $\left\{ \frac{1}{n^2\sqrt{n}} \right\}$ ① $a_{n+1} \le a_n$ ② $\lim_{n\to\infty} a_n = 0$

① $\frac{1}{(n+1)^2\sqrt{n+1}} \le \frac{1}{n^2\sqrt{n}}$ ✓

② $\lim_{n\to\infty} \frac{1}{n^2\sqrt{n}} = 0$ ✓

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\sqrt{n}}$ converges.

Test for absolute convergence, Look at

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2\sqrt{n}}$$

converges p-series $p = \frac{5}{2} > 1$

$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\sqrt{n}} \text{ converges absolutely}}$$

b.) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

First test for absolute convergence:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$ divergent p-series.

$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ does not converge absolutely

But $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ does converge by AST since $\left\{\frac{1}{\sqrt{n}}\right\}$ decreases to zero $\therefore \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges conditionally.

c.) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$

Absolute convergence?

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

$\frac{1}{n(\ln n)^2} > 0$ & decreases

easy to integrate, use I.T.

$u = \ln x$

$du = \frac{dx}{x}$

$\int \frac{du}{u^2} = \frac{-1}{u}$

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = -\frac{1}{\ln x}\Bigg|_2^{\infty}$$

$$= \cancel{-\frac{1}{\ln \infty}}^{\to 0} + \frac{1}{\ln 2}$$

$$= \frac{1}{\ln 2} < \infty$$

integral converges, so $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges,

$$\boxed{\text{so } \sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2} \text{ converges absolutely}}$$

d.) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+1}$

TD $\lim_{n \to \infty} \frac{(-1)^n n}{n+1} \neq 0$

$$\therefore \boxed{\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+1} \text{ diverges}}$$

Ratio Test: If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} <1 & \text{converges absolutely} \\ >1 & \text{diverges} \\ =1 & \text{test fails} \end{cases}$

$n! \to$ Ratio Test

$a^n \to$ Ratio Test

$\hookrightarrow 2^n,\ \left(\frac{1}{4}\right)^n,\ e^n,$ etc

e.) $\sum_{n=1}^{\infty} \frac{n^2}{(-4)^n}$

RT: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{(-4)^{n+1}}\cdot\frac{(-4)^n}{n^2}\right|$

$= \lim_{n\to\infty}\left|\frac{(n+1)^2}{(-4)\cancel{(-4)^n}}\cdot\frac{\cancel{(-4)^n}}{n^2}\right|$

$= \lim_{n\to\infty}\left|\frac{(n+1)^2}{-4n^2}\right|$

$= \left|-\frac{1}{4}\right| = \frac{1}{4} < 1$

converges absolutely

f.) $\sum_{n=1}^{\infty} \frac{3^n n^2}{(2n)!}$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}(n+1)^2}{(2n+2)!}\cdot\frac{(2n)!}{3^n n^2}\right|$

$= \lim_{n\to\infty}\left|\frac{3\cdot\cancel{3^n}(n+1)^2}{(2n+2)(2n+1)\cancel{(2n)!}}\cdot\frac{\cancel{(2n)!}}{\cancel{3^n}n^2}\right|$

$= \lim_{n\to\infty}\left|\frac{3(n+1)^2}{(2n+2)(2n+1)n^2}\right| = 0 < 1$

converges absolutely

6. Show $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$ converges absolutely and then approximate the sum of the series with the third partial sum, s_2. How close is this approximation to the sum of the series?

If $\sum_{n=1}^{\infty} (-1)^n a_n$ is a convergent alternating series

Then $|R_n| = |S - S_n| \leq a_{n+1}$

$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$ converges absolutely by ratio test.

$\boxed{S_2 = 1 - \frac{1}{3!} + \frac{1}{5!}}$

$|R_2| \leq |a_3| = \frac{1}{7!}$

7. Approximate $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ correct to within 3 decimal places.

since $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is an alternating series,

use $|R_n| \leq |a_{n+1}| = \frac{1}{(n+1)^2} \leq \frac{1}{10^3}$

$10^3 \leq (n+1)^2$

$\sqrt{10^3} \leq n+1$

$31.6 \leq n+1$

$30.6 \leq n$

n must be at least 31

use S_{31} to approximate the sum