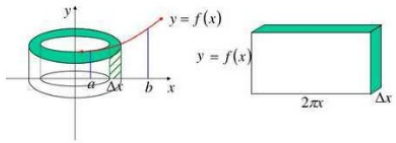
Section 6.3: Volume by Cyndrical Shells

The Method of Cylindrical Shells: Within the bounded region, we rotate a rectangle around the axis of rotation. This results in what is called a **cylindrical shell**:

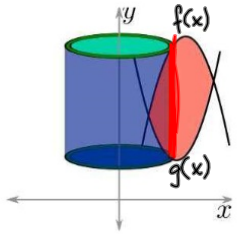


$$V_i = 2\pi x \cdot f(x) \cdot dx$$

$$V = \int_a^b 2\pi x \cdot f(x)\, dx$$

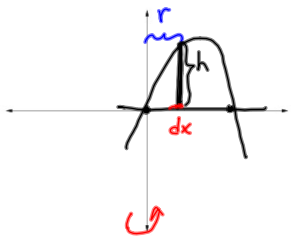
- **Revolution about the y axis:** $V = \int_a^b 2\pi x(f(x) - g(x))\, dx$,

where $f(x) \geq g(x)$ for $a \leq x \leq b$.



$$V_i = 2\pi x \left(f(x) - g(x)\right) dx$$

1. Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$, $y = 0$, about the y axis.

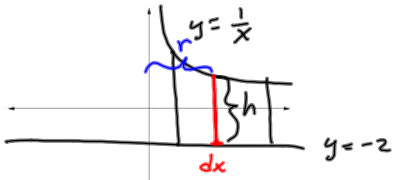


$$x - x^2 = 0$$
$$x(1-x) = 0$$
$$x = 0 \qquad x = 1$$

$$r = x - 0$$
$$h = x - x^2 - 0$$

$$V = \int_0^1 2\pi x \cdot (x - x^2)\, dx = 2\pi \int_0^1 x^2 - x^3\, dx$$
$$= 2\pi \left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]\Big|_0^1 = \frac{2\pi}{12} = \boxed{\frac{\pi}{6}}$$

2. Find the volume of the solid obtained by rotating the region bounded by $y = \frac{1}{x}$, $y = -2$, $x = 1$, $x = 4$ about the y axis.



$r = x - 0$

$h = \frac{1}{x} - (-2) = \frac{1}{x} + 2$

$$V = 2\pi \int_1^4 x \left(\frac{1}{x} + 2\right) dx$$

$$V = 2\pi \int_1^4 1 + 2x \, dx = 2\pi \left[x + x^2\right]\Big|_1^4 = \boxed{36\pi}$$

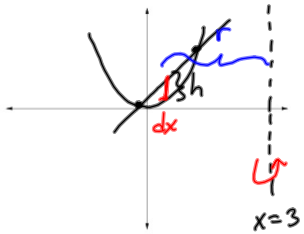
3. Set up but do not evaluate an integral that gives the volume of the solid obtained by rotating the region bounded by $y = x^2$, $y = x$ about the line $x = 3$ using two different methods.

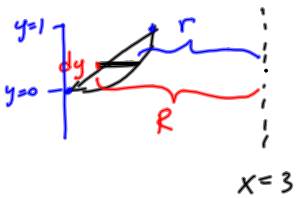
$x^2 = x$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0 \quad x = 1$





$y = x^2$

$x = \pm\sqrt{y}$

$x = y$

$r = 3 - x$

$h = x - x^2$

$$\boxed{V = 2\pi \int_0^1 (3-x)(x-x^2)\, dx}$$

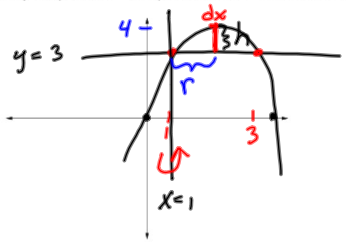
Shell Method

$R = 3 - y$

$r = 3 - \sqrt{y}$

$$\boxed{V = \pi \int_0^1 (3-y)^2 - (3-\sqrt{y})^2 \, dy}$$

4. Set up but do not evaluate an integral that gives the volume of the solid obtained by rotating the region $y = 4x - x^2$, $y = 3$, about the line $x = 1$.



$4x - x^2 = 0$

$x(4-x) = 0$

$x = 0 \quad x = 4$

$4x - x^2 = 3$

$0 = x^2 - 4x + 3$

$0 = (x-3)(x-1)$

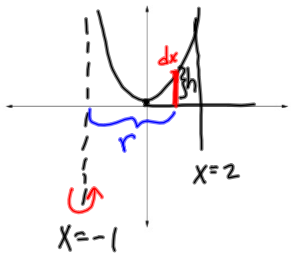
$x = 1 \quad x = 3$

$r = x - 1$

$h = 4x - x^2 - 3$

$$\boxed{V = 2\pi \int_1^3 (x-1)(4x - x^2 - 3)\, dx}$$

5. Set up but do not evaluate an integral that gives volume of the solid obtained by rotating the region bounded by $y = 3x^2$, $y = 0$, $x = 0$, $x = 2$ about the line $x = -1$.



$r = x - (-1) = x + 1$

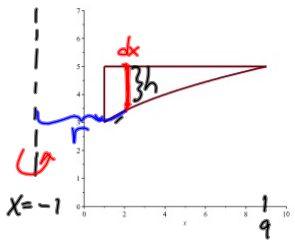
$h = 3x^2 - 0$

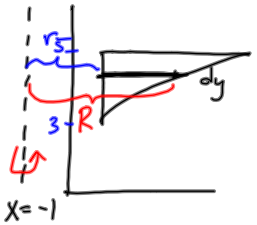
$$\boxed{V = 2\pi \int_0^2 (x+1)(3x^2)\, dx}$$

$5 = \sqrt{x} + 2 \quad x = 9$

$3 = \sqrt{x}$

6. Set up but do not evaluate an integral that gives the volume of the solid obtained by rotating the region $y = \sqrt{x} + 2$, $x = 1$, $y = 5$, about the line $x = -1$ using two different methods.





$y = \sqrt{x} + 2$

$y - 2 = \sqrt{x}$

$x = (y-2)^2$

$R = (y-2)^2 - (-1)$

$= (y-2)^2 + 1$

$r = x - (-1) = x + 1$

$r = 1 - (-1) = 2$

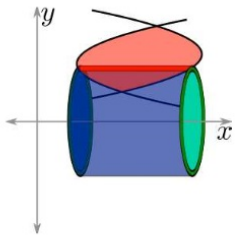
$h = 5 - (\sqrt{x} + 2) = 3 - \sqrt{x}$

$$\boxed{V = 2\pi \int_1^9 (x+1)(3 - \sqrt{x})\, dx}$$

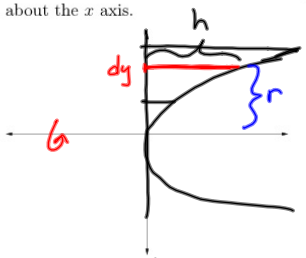
$$\boxed{V = \pi \int_3^5 \left[(y-2)^2 + 1\right]^2 - 2^2 \, dy}$$

- **Revolution about the x axis:** $V = \int_c^d 2\pi y(f(y) - g(y))\, dy$,

where $f(y) \geq g(y)$ for $c \leq y \leq d$.



7. Find the volume of the solid obtained by rotating the region bounded by $y^2 = x$, $x = 0$, $y = 2$, $y = 5$ about the x axis.



$r = y - 0$

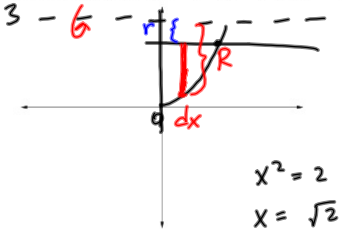
$h = y^2 - 0$

$$V = 2\pi \int_2^5 y \cdot y^2 \, dy = 2\pi \int_2^5 y^3 \, dy$$

$$2\pi \cdot \frac{1}{4} y^4 \Big|_2^5 = \frac{\pi}{2}[625 - 16] = \boxed{\frac{\pi}{2} \cdot 609}$$

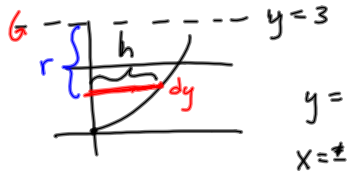
8. Set up but do not evaluate an integral that gives the volume of the solid obtained by rotating the region bounded by $y = x^2$, $x = 0$ and $y = 2$ about the line $y = 3$ using two different methods.

$y = 3$ — 6 — r — — — — in the first quadrant



$x^2 = 2$

$x = \sqrt{2}$



$y = x^2$

$x = \pm\sqrt{y}$

$R = 3 - x^2$

$r = 3 - 2 = 1$

$r = 3 - y$

$h = \sqrt{y} - 0$

$$\boxed{V = \pi \int_0^{\sqrt{2}} (3 - x^2)^2 - 1^2 \, dx}$$

Washer Method

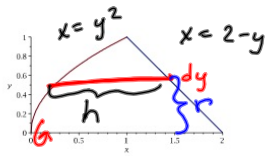
$$\boxed{V = 2\pi \int_0^2 (3 - y)(\sqrt{y}) \, dy}$$

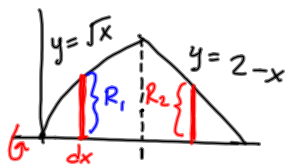
Shell Method

$y = \sqrt{x}$ $\qquad$ $x + y = 2$

$x = y^2$ $\qquad$ $x = 2 - y$

9. Using two different methods, find an integral that gives the volume of the solid obtained by rotating region bounded by $y = \sqrt{x}$, $y = 0$, $x + y = 2$ about the x-axis. Do not evaluate either integral.





$r = y - 0$

$h = (2-y) - (y^2)$

$$\boxed{V = 2\pi \int_0^1 y\,(2 - y - y^2)\,dy}$$

Shell Method

$R_1 = \sqrt{x}$ $\qquad$ $R_2 = 2 - x$

$$\boxed{V = \pi \int_0^1 (\sqrt{x})^2\,dx + \pi \int_1^2 (2-x)^2\,dx}$$

Disk Method