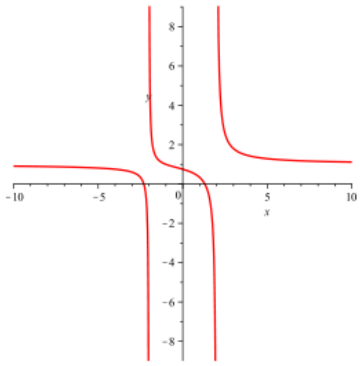
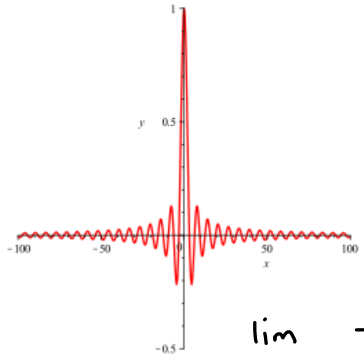
Section 2.6: Limits at Infinity; Horizontal Asymptotes

Definition If $\lim_{x\to\infty} f(x) = L$, or if $\lim_{x\to-\infty} f(x) = L$, then we say $f(x)$ has a **horizontal asymptote** at $\boxed{y = L.}$





$$\lim_{x\to\infty} \frac{1}{x} = 0$$

EXAMPLE 1: Find the limits:

$\frac{4x}{x^3} = \frac{4}{x^2}$

a.) $$\lim_{x\to\infty} \frac{7x^3 + 4x}{2x^3 - x^2 + 3} = \lim_{x\to\infty} \frac{\cancel{x^3}\left(7 + \frac{4}{x^2}\right)}{\cancel{x^3}\left(2 - \frac{1}{x} + \frac{3}{x^3}\right)} = \frac{7}{2}$$

b.) $$\lim_{t\to\infty} \frac{t^4 - t^2 + 1}{t^5 + t^3 - t} = \lim_{t\to\infty} \frac{\cancel{t^4}\left(1 - \frac{1}{t^2} + \frac{1}{t^4}\right)}{\overset{t}{\cancel{t^5}}\left(1 + \frac{1}{t^2} - \frac{1}{t^4}\right)}$$

$$= \lim_{t\to\infty} \frac{1 - \frac{1}{t^2} + \frac{1}{t^4}}{t\left(1 + \frac{1}{t^2} - \frac{1}{t^4}\right)}$$

$$= \frac{1}{\infty}$$

$$= \boxed{0}$$

c.) $\lim_{x\to-\infty} \frac{x^4+2x+3}{x(x^2-1)} = \lim_{x\to-\infty} \frac{x^4+2x+3}{x^3-x}$

$$= \lim_{x\to-\infty} \frac{x^4\left(1+\cancelto{0}{\frac{2}{x^3}}+\cancelto{0}{\frac{3}{x^4}}\right)}{x^3\left(1-\cancelto{0}{\frac{1}{x^2}}\right)}$$

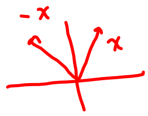
$$= \lim_{x\to-\infty} \frac{x^4}{x^3} = \lim_{x\to-\infty} x$$

$$= \boxed{-\infty}$$

d.) $\lim_{x\to\infty} \frac{\sqrt{1+4x^2}}{4+x} = \lim_{x\to\infty} \frac{\sqrt{x^2\left(\frac{1}{x^2}+4\right)}}{x\left(\frac{4}{x}+1\right)}$

Recall:

$$\sqrt{x^2} = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$



$$= \lim_{x\to\infty} \frac{\sqrt{x^2}\sqrt{\frac{1}{x^2}+4}}{x\left(\frac{4}{x}+1\right)}$$

since $x \to +\infty$, $x > 0$, $|x| = x$

$$= \lim_{x\to\infty} \frac{|x|\sqrt{\frac{1}{x^2}+4}}{x\left(\frac{4}{x}+1\right)}$$

$$= \lim_{x\to\infty} \frac{\cancel{x}\sqrt{\cancelto{0}{\frac{1}{x^2}}+4}}{\cancel{x}\left(\cancelto{0}{\frac{4}{x}}+1\right)} = \frac{\sqrt{4}}{1}$$

$$= \boxed{2}$$

e.) $\lim_{x\to-\infty} \frac{\sqrt{x^2+4x}}{4x+1} = \lim_{x\to-\infty} \frac{\sqrt{x^2\left(1+\frac{4}{x}\right)}}{x\left(4+\frac{1}{x}\right)}$

$$= \lim_{x\to-\infty} \frac{\sqrt{x^2}\sqrt{1+\frac{4}{x}}}{x\left(4+\frac{1}{x}\right)}$$

$$= \lim_{x\to-\infty} \frac{|x|\sqrt{1+\frac{4}{x}}}{x\left(4+\frac{1}{x}\right)}$$

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

since $x < 0$, $|x| = -x$

$$= \lim_{x\to-\infty} \frac{-\cancel{x}\sqrt{1+\frac{4}{x}}}{\cancel{x}\left(4+\frac{1}{x}\right)}$$

$$= \lim_{x\to-\infty} -\frac{\sqrt{1+\cancelto{0}{\frac{4}{x}}}}{4+\cancelto{0}{\frac{1}{x}}}$$

$$= \boxed{-\frac{1}{4}}$$

f.) $\lim_{x\to\infty}(\sqrt{x^2+3x+1}-x)\dfrac{\left(\sqrt{x^2+3x+1}+x\right)}{\sqrt{x^2+3x+1}+x}$

$$= \lim_{x\to\infty}\frac{\cancel{x^2}+3x+1-\cancel{x^2}}{\sqrt{x^2+3x+1}+x}$$

$$\to \lim_{x\to\infty}\frac{x\left(3+\frac{1}{x}\right)}{\sqrt{x^2\left(1+\frac{3}{x}+\frac{1}{x^2}\right)}+x}$$

$$= \lim_{x\to\infty}\frac{x\left(3+\frac{1}{x}\right)}{|x|\sqrt{1+\frac{3}{x}+\frac{1}{x^2}}+x}$$

$$= \lim_{x\to\infty}\frac{x\left(3+\frac{1}{x}\right)}{x\sqrt{1+\frac{3}{x}+\frac{1}{x^2}}+x}$$

$$= \lim_{x\to\infty}\frac{\cancel{x}\left(3+\cancel{\frac{1}{x}}^{0}\right)}{\cancel{x}\left[\sqrt{1+\cancel{\frac{3}{x}}^{0}+\cancel{\frac{1}{x^2}}^{0}}+1\right]}$$

$$= \frac{3}{\sqrt{1}+1} = \boxed{\frac{3}{2}}$$

g.) $\lim_{x\to-\infty}(x+\sqrt{x^2+2x})\dfrac{\left(x-\sqrt{x^2+2x}\right)}{x-\sqrt{x^2+2x}}$

$$\lim_{x\to-\infty}\frac{\cancel{x^2}-\left(\cancel{x^2}+2x\right)}{x-\sqrt{x^2+2x}}$$

$$\to = \lim_{x\to-\infty}\frac{-2x}{x-\sqrt{x^2+2x}}$$

$$= \lim_{x\to-\infty}\frac{-2x}{x-\sqrt{x^2\left(1+\frac{2}{x}\right)}}$$

$$= \lim_{x\to-\infty}\frac{-2x}{x-|x|\sqrt{1+\frac{2}{x}}}$$

since $x\to-\infty$,
$x<0$, $|x|=-x$

$$= \lim_{x\to-\infty}\frac{-2x}{x-(-x)\sqrt{1+\frac{2}{x}}}$$

$$= \lim_{x\to-\infty}\frac{-2x}{x+x\sqrt{1+\frac{2}{x}}}$$

$$= \lim_{x\to-\infty}\frac{-2\cancel{x}}{\cancel{x}\left[1+\sqrt{1+\cancel{\frac{2}{x}}^{0}}\right]}$$

$$= \frac{-2}{1+\sqrt{1}} = \boxed{-1}$$