Section 4.2: Inverse Functions

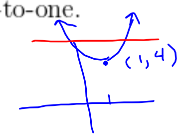
Definition: We say $f(x)$ is one-to-one provided whenever $f(x_1) = f(x_2)$, $x_1 = x_2$.

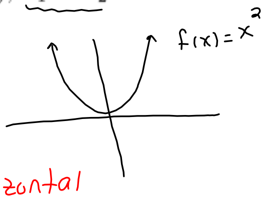
EXAMPLE 1: Prove $f(x) = x^2 - 2x + 5$ is not one-to-one.

vertex: $f'(x) = 0$

$2x - 2 = 0$

$x = 1$, $y = 1 - 2 + 5 = 4$





Fails horizontal line test.

$f(0) = 5$

$f(2) = 4 - 4 + 5 = 5$

$f(-2) = 4$

$f(2) = 4$

$f(-2) = f(2)$

but $-2 \neq 2$

EXAMPLE 2: Prove $f(x) = 5 - 4x^3$ is one-to-one.

Assume $f(x_1) = f(x_2)$ [deduce $x_1 = x_2$]

$\cancel{5} - \cancel{4}x_1^3 = \cancel{5} - \cancel{4}x_2^3$

$x_1^3 = x_2^3$

$\sqrt[3]{x_1^3} = \sqrt[3]{x_2^3} \rightarrow x_1 = x_2 \leftarrow$ one solution because of odd power

$f(x) = x^2$

$f(x_1) = f(x_2)$

$x_1^2 = x_2^2$

$\pm x_1 = \pm x_2$

doesn't work for even powers

EXAMPLE 3: Prove $f(x) = \dfrac{x-2}{x+2}$ is one-to-one.

$f(x_1) = f(x_2)$

$\dfrac{x_1 - 2}{x_1 + 2} = \dfrac{x_2 - 2}{x_2 + 2}$

$(x_1 - 2)(x_2 + 2) = (x_2 - 2)(x_1 + 2)$

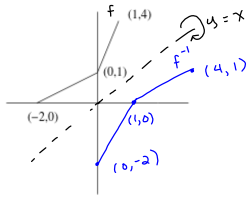
$\cancel{x_1 x_2} + 2x_1 - 2x_2 - \cancel{4} = \cancel{x_2 x_1} + 2x_2 - 2x_1 - \cancel{4}$

$\cancel{4}x_1 = \cancel{4}x_2$ $\boxed{x_1 = x_2}$

Definition: Let $f(x)$ be a one-to-one function with domain D and range R. Then the inverse exists, denoted by $f^{-1}(x)$. Furthermore, the domain of f^{-1} = range of $f = R$ and the range of f^{-1} = domain of $f = D$. Moreover,

$$f(x) = y \iff f^{-1}(y) = x$$

EXAMPLE 4: Given the graph of f below, sketch the graph of f^{-1}.



EXAMPLE 5: Find the inverse, Find domain & range of f^{-1}.

(a) $f(x) = 5 - 4x^3$

$x \to \infty \quad f \to -\infty$

$x \to -\infty \quad f \to \infty$

$$\boxed{\begin{aligned} &\text{domain } f : (-\infty, \infty) = \text{range } f^{-1} \\ &\text{range } f : (-\infty, \infty) = \text{domain } f^{-1} \end{aligned}}$$

Find $f^{-1}(x)$: $y = 5 - 4x^3$

$x = 5 - 4y^3$

$4y^3 = 5 - x$

$y^3 = \dfrac{5-x}{4}$

$y = \sqrt[3]{\dfrac{5-x}{4}}$

$$\boxed{f^{-1}(x) = \sqrt[3]{\frac{5-x}{4}}}$$

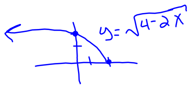
(b) $f(x) = \sqrt{4 - 2x}$

$$\boxed{\begin{aligned} &\text{domain } f = (-\infty, 2] = \text{range } f^{-1} \\ &\text{range } f = [0, \infty) = \text{domain } f^{-1} \end{aligned}}$$

$4 - 2x \geq 0$

$4 \geq 2x$

$2 \geq x$



Find $f^{-1}(x)$: $y = \sqrt{4-2x}$

$x = \sqrt{4-2y}$

$x^2 = 4 - 2y$

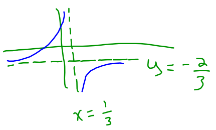
$y = \dfrac{4-x^2}{2}$

$$\boxed{f^{-1}(x) = \frac{4-x^2}{2}}$$

(c) $f(x) = \dfrac{2x+1}{1-3x}$

$$\boxed{\begin{aligned} &\text{domain } f : \left(-\infty, \tfrac{1}{3}\right) \cup \left(\tfrac{1}{3}, \infty\right) = \text{range } f^{-1} \\ &\text{range } f : \left(-\infty, -\tfrac{2}{3}\right) \cup \left(-\tfrac{2}{3}, \infty\right) = \text{domain } f^{-1} \end{aligned}}$$

$\lim\limits_{x \to \pm\infty} \dfrac{2x+1}{1-3x} = -\dfrac{2}{3}$



Find $f^{-1}(x)$: $y = \dfrac{2x+1}{1-3x}$

$x = \dfrac{2y+1}{1-3y}$

$x(1-3y) = 2y+1$

$x - 3xy = 2y + 1$

$x - 1 = 2y + 3xy$

$x - 1 = y(2+3x)$

$y = \boxed{\dfrac{x-1}{2+3x} = f^{-1}(x)}$

Theorem: Suppose f is a one-to-one differentiable function with inverse function $g = f^{-1}$. Then g is differentiable and $\boxed{g'(a) = \dfrac{1}{f'(g(a))}}$.

EXAMPLE 6: Suppose g is the inverse of f and $f(2) = 3$, $f'(2) = 7$, $f(3) = 4$ and $f'(3) = \frac{1}{2}$. $\boxed{\text{Find } g'(3).}$

$f^{-1}(3) = 2$

$(f^{-1})'(3) = \dfrac{1}{f'(f^{-1}(3))}$

$g'(3) = \dfrac{1}{f'(g(3))}$

$g(3)$ = number that f maps to 3

$g'(3) = \dfrac{1}{f'(g(3))}$

$\boxed{g(3) = 2}$ because $f(2) = 3$

$= \dfrac{1}{f'(2)} = \boxed{\dfrac{1}{7}}$

EXAMPLE 7: Suppose g is the inverse of f. Find $g'(4)$ if $f(x) = 3 + x + e^x$.

$g'(4) = \dfrac{1}{f'(g(4))}$

$g(4) = f^{-1}(4)$

solution to $3 + x + e^x = 4$

$x = 0$, meaning $f(0) = 4$

thus $f^{-1}(4) = 0$

$g(4) = 0$

$f(x) = 3 + x + e^x$

$f'(x) = 1 + e^x$

$f'(0) = 2$

$= \dfrac{1}{f'(0)}$

$= \dfrac{1}{2}$

EXAMPLE 8: Suppose g is the inverse of f. Find $g'(2)$ if $f(x) = \sqrt{x^3 + x^2 + x + 1}$.

$g'(2) = \dfrac{1}{f'(g(2))}$

solution to $\sqrt{x^3 + x^2 + x + 1} = 2$

$x = 1$

meaning $f(1) = 2 \longrightarrow f^{-1}(2) = 1$

$\longrightarrow g(2) = 1$

$= \dfrac{1}{f'(1)}$

$\boxed{g'(2) = \dfrac{2}{3}}$

$f(x) = (x^3 + x^2 + x + 1)^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}(x^3 + x^2 + x + 1)^{-\frac{1}{2}}(3x^2 + 2x + 1)$

$f'(1) = \frac{1}{2}(4)^{-\frac{1}{2}}(6)$

$= \frac{1}{2} \cdot \frac{1}{4^{1/2}} \cdot 6$

$= \frac{1}{4} \cdot 6$

$= \frac{3}{2}$