Section 4.6: Inverse Trigonometric Functions

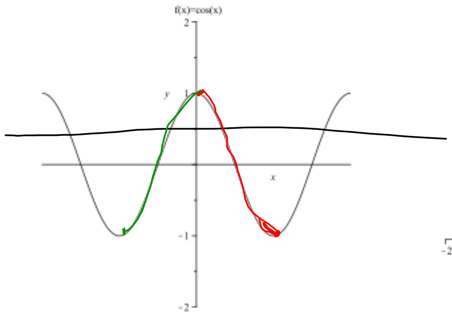
I. INVERSE COSINE: If $0 \leq x \leq \pi$, then $f(x) = \cos x$ is one-to-one, thus the inverse exists, denoted by $\cos^{-1}(x)$ or $\arccos x$. Additionally, the domain of $\arccos x$ = range of $\cos x = [-1, 1]$ and range of $\arccos x$ = domain of $\cos x = [0, \pi]$. Note: $\arccos(x)$ is the **angle** in $[0, \pi]$ whose cosine is x.

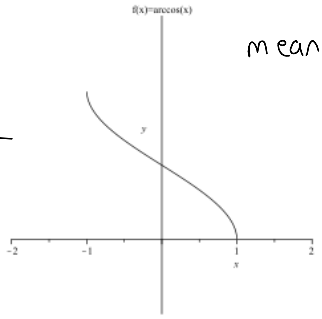
$\arccos(r) = \theta$

means $\cos\theta = r$

$\boxed{0 \leq \theta \leq \pi}$

θ must be in Q I or II





Cancellation Equations: Recall $f^{-1}(f(x)) = x$ for x in the domain of f, and $f(f^{-1}(x)) = x$ for x in the domain of f^{-1}. This yields the following cancellation equations:

- $\arccos(\cos x) = x$ if $0 \leq x \leq \pi$
- $\cos(\arccos x) = x$ if $-1 \leq x \leq 1$.

Example 1: Compute the following.

(i) $\arccos(0) = \theta$

$\cos\theta = 0$, $0 \leq \theta \leq \pi$

$\boxed{\theta = \frac{\pi}{2}}$

(ii) $\cos^{-1}(1) = \theta$

$\cos\theta = 1$ $\quad 0 \leq \theta \leq \pi$

$\boxed{\theta = 0}$

(iii) $\arccos(-1) = \theta$

$\cos\theta = -1$

$\boxed{\theta = \pi}$

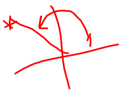
(iv) $\arccos\frac{1}{2} = \theta$

$\cos\theta = \frac{1}{2}$

$\boxed{\theta = \frac{\pi}{3}}$

(v) $\cos^{-1}\frac{-\sqrt{3}}{2} = \Theta$

$\cos\Theta = -\frac{\sqrt{3}}{2}$



$\Theta = \pi - \frac{\pi}{6}$

$\Theta = \frac{5\pi}{6}$

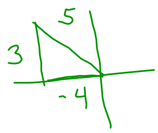
$f^{-1}(f(x)) = x$

$f(f^{-1}(x)) = x$

(vi) $\sin\left(2\overset{\Theta}{\arccos(-\frac{4}{5})}\right) = \sin(2\Theta) = 2\sin\Theta\cos\Theta$

$\Theta = \arccos(-\frac{4}{5})$

$\cos\Theta = -\frac{4}{5} = \frac{adj}{hyp}$



$\sin(2\Theta) = 2\sin\Theta\cos\Theta$

$= \boxed{2(\frac{3}{5})(-\frac{4}{5})}$

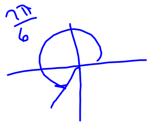
(vii) $\arccos\left((\cos(\frac{\pi}{6})\right)$

$\boxed{\Theta = \frac{\pi}{6}}$

$\arccos(\cos\frac{\pi}{6}) = \arccos(\frac{\sqrt{3}}{2})$

$\Theta = \frac{\pi}{6}$

(viii) $\arccos\left((\cos(\frac{7\pi}{6})\right) = \underline{\underline{\frac{5\pi}{6}}}$



$\arccos(-\frac{\sqrt{3}}{2}) = \Theta$

$\cos\Theta = -\frac{\sqrt{3}}{2}$

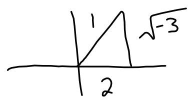
$\boxed{\Theta = \frac{5\pi}{6}}$

(ix) $\arccos\left((\cos(-\frac{\pi}{3})\right) = \frac{\pi}{3}$

(x) $\cos((\underline{\cos^{-1}}(2))$

~~$\Theta = \cos^{-1}(2)$~~

~~$\cos\Theta = 2$~~



dne because domain arccos x

= range cos x

= [-1, 1]

<u>DNE</u>

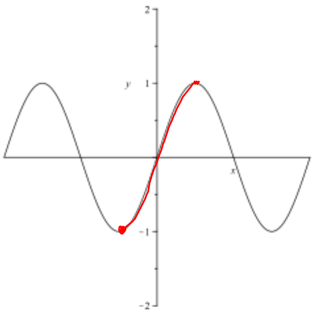
II. INVERSE SINE: If $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, then $f(x) = \sin x$ is one-to-one, thus the inverse exists, denoted by $\sin^{-1}(x)$ or $\arcsin x$. Additionally, the domain of $\arcsin x =$ range of $\sin x = [-1, 1]$ and range of $\arcsin x =$ domain of $\sin x = [-\frac{\pi}{2}, \frac{\pi}{2}]$.
Note: $\arcsin(x)$ is the **angle** in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ whose sine is x.

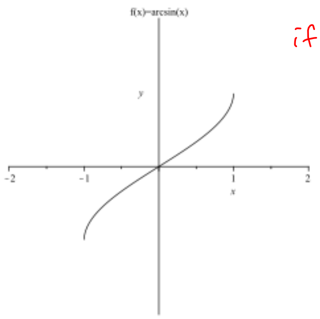
$\arcsin(r) = \theta$

if $\sin\theta = r$

$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

I or $\underline{\text{IV}}$





Cancellation Equations: Recall $f^{-1}(f(x)) = x$ for x in the domain of f, and $f(f^{-1}(x)) = x$ for x in the domain of f^{-1}. This yields the following cancellation equations:

- $\arcsin(\sin x) = x$ if $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$
- $\sin(\arcsin x) = x$ if $-1 \leq x \leq 1$.

Example 3: Compute the following.

(i) $\arcsin(0) = \theta$

$\sin\theta = 0$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$\boxed{\theta = 0}$

(ii) $\sin^{-1}(1) = \theta$

$\sin\theta = 1$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$\boxed{\theta = \frac{\pi}{2}}$

(iii) $\arcsin(-1) = \theta$

$\sin\theta = -1$

$\boxed{\theta = -\frac{\pi}{2}}$

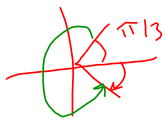
(iv) $\arcsin\frac{1}{2} = \theta$

$\sin\theta = \frac{1}{2}$

$\boxed{\theta = \frac{\pi}{6}}$

(v) $\sin^{-1} \frac{-\sqrt{3}}{2} = \theta$

$\sin\theta = -\frac{\sqrt{3}}{2}$

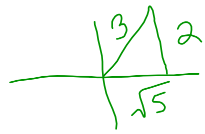


$\theta = \frac{-\pi}{3}$

(vi) $\overset{*}{\tan} \left(\underbrace{\arcsin(\frac{2}{3})}_{\theta} \right)$

$\theta = \arcsin \frac{2}{3}$

$\sin\theta = \frac{2}{3}$



$\tan\theta = \frac{2}{\sqrt{5}}$

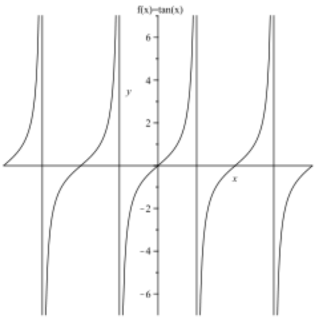
(vii) $\sin \left(\arcsin(\frac{3}{10}) \right)$

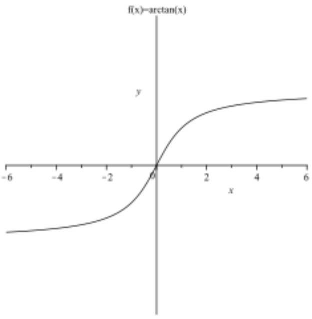
(viii) $\arcsin \left((\sin(\frac{5\pi}{4}) \right)$

(ix) $\arcsin \left((\sin(-\frac{\pi}{6}) \right)$

(x) $\arcsin \left(\sin \frac{\pi}{120} \right)$

III. INVERSE TANGENT: If $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$, then $f(x) = \tan x$ is one-to-one, thus the inverse exists, denoted by $\tan^{-1}(x)$ or $\arctan x$. Additionally, the domain of $\arctan x =$ range of $\tan x = (-\infty, \infty)$ and range of $\arctan x =$ domain of $\tan x = (-\dfrac{\pi}{2}, \dfrac{\pi}{2})$. Note: $\arctan(x)$ is the **angle** in $(-\dfrac{\pi}{2}, \dfrac{\pi}{2})$ whose tangent is x.





Cancellation Equations: Recall $f^{-1}(f(x)) = x$ for x in the domain of f, and $f(f^{-1}(x)) = x$ for x in the domain of f^{-1}. This yields the following cancellation equations:

- $\arctan(\tan x) = x$ if $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$
- $\tan(\arctan x) = x$ for all x.

Example 5: Compute the following.

(i) $\arctan(0)$ (ii) $\tan^{-1}(1)$

(iii) $\arctan(-1)$ (iv) $\arctan(-\sqrt{3})$

(v) $\tan(\arcsin x)$

(vi) $\arctan\left(\tan(\frac{5\pi}{3})\right)$

(vii) $\lim_{x\to\infty} \arctan x$

(viii) $\lim_{x\to-\infty} \arctan x$

Derivatives of Inverse Trigonometric Functions:

A.) $\dfrac{d}{dx} \arcsin x = \dfrac{1}{\sqrt{1-x^2}}$.

B.) $\dfrac{d}{dx} \arccos x = -\dfrac{1}{\sqrt{1-x^2}}$.

C.) $\dfrac{d}{dx} \arctan x = \dfrac{1}{1+x^2}$.

Example 7: Prove formula A.

Example 8: Find the derivative of $f(x) = \arccos(2x - 1)$.

Example 9: Find the derivative of $f(x) = \tan^{-1}(\arcsin x)$.

Example 10: What is the domain of $\arcsin(3x + 1)$? Of $\arctan(3x + 1)$?

The Unit Circle