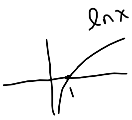
Section 4.8: L'Hospital's Rule

Indeterminate form: If $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$, then we say the limit is in indeterminate form.

L'Hospital's Rule: If $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.

Some common misconceptions: If If $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{\infty}$ or $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\infty}{0}$, the limit is NOT indeterminate! For example,



(i) $\lim_{x\to 0^+} \frac{\ln x}{\sqrt{x}} = \frac{-\infty}{0^+} = (-\infty)\left(\frac{1}{0^+}\right)$

$= (-\infty)(\infty)$

$= -\infty$

(ii) $\lim_{x\to 0^+} \frac{x}{\ln x} = \frac{0^+}{-\infty}$

$= 0^+\left(\frac{1}{-\infty}\right)$ 0^-

$= 0$

Example 1: Find the following limits, if they exist. If the limit does not exist, explain why.

(i) $\lim_{x\to 1} \frac{\ln x}{x-1} = \frac{0}{0}$

$\overset{L}{=} \lim_{x\to 1} \frac{\frac{1}{x}}{1} = \boxed{1}$

(ii) $\lim_{x\to 0} \frac{\sin x - x}{x^3} = \frac{0}{0}$

$\overset{L}{=} \lim_{x\to 0} \frac{\cos x - 1}{3x^2}$ $\frac{0}{0}$

$\overset{L}{=} \lim_{x\to 0} \frac{-\sin x}{6x}$ $\frac{0}{0}$ $\quad 1 = -\frac{1}{6}$

$\overset{L}{=} \lim_{x\to 0} \frac{-\cos x}{6} = -\frac{1}{6}$

(iii) $\lim_{x\to 0} \frac{\sin mx}{\sin nx}$ $\frac{0}{0} \overset{L}{=} \lim_{x\to 0} \frac{m\cos(mx)}{n\cos(nx)} = \frac{m}{n}$

$\ln x$

(iv) $\lim_{x\to\infty} \frac{(\ln x)^2}{x}$ $= \frac{\infty}{\infty} \overset{L}{=} \lim_{x\to\infty} \frac{2(\ln x)(\frac{1}{x})}{1}$

$= \lim_{x\to\infty} \frac{2\ln x}{x} = \frac{\infty}{\infty} \overset{L}{=} \lim_{x\to\infty} \frac{2\cdot\frac{1}{x}}{1}$

$(.0000001)(100)$ $(.01)(10000)$ $= \lim_{x\to\infty} \frac{2}{x} = \boxed{0}$

Indeterminate Products: If $\lim_{x\to a} f(x)g(x) = 0\cdot\infty$, this limit is an indeterminate product. Why do we call the product indeterminate?

$\lim_{x\to\infty} \frac{1}{x^2}\cdot x = 0\cdot\infty = 0$ $\qquad$ $\lim_{x\to\infty} \frac{1}{x}\cdot x^2 = 0\cdot\infty = \infty$ $\qquad$ $\lim_{x\to\infty} \frac{1}{x^2}\cdot 6x^2 = 0\cdot\infty = 6$

All three of these limits are of the form $0\cdot\infty$, yet they all have different limits. The goal is ty try to manipulate the product get the limit in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then use L'Hospital's rule.

Example 2: Find the following limits, if they exist. If the limit does not exist, explain why.

(i) $\lim_{x\to 0^+} x^3 \ln x = 0(-\infty)$ $\qquad$ "Bring one down"

$\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x^3}} = \frac{-\infty}{\frac{1}{0}=\infty}$ $\qquad$ OR $\qquad$ $\cancel{\lim_{x\to 0^+} \frac{x^3}{\frac{1}{\ln x}} = \frac{0}{\frac{1}{-\infty}=0}}$

$\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x^3}}$ $\frac{-\infty}{\infty} \overset{L}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{3}{x^4}} = \lim_{x\to 0^+} \frac{1}{x}\left(-\frac{x^4}{3}\right)$

$= \lim_{x\to 0^+} -\frac{x^3}{3} = \boxed{0}$

(ii) $\lim_{x\to 1^+} (x-1)\tan(\pi x/2) = (0)(\tan\frac{\pi}{2})$ $\qquad$ "Bring one down"

$= (0)(-\infty)$

$\cancel{\lim_{x\to 1^+} \frac{\tan(\pi\frac{x}{2})}{\frac{1}{x-1}}}$ $\qquad$ OR $\qquad$ * $\lim_{x\to 1^+} \frac{x-1}{\frac{1}{\tan\frac{\pi x}{2}}}$ *

$\lim_{x\to 1^+} \frac{x-1}{\cot\frac{\pi x}{2}}$ $\quad \frac{0}{0}$

$\csc(\frac{\pi}{2}) = \frac{1}{\sin\frac{\pi}{2}}$

$= 1$ $\qquad$ $\overset{L}{=} \lim_{x\to 1^+} \frac{1}{-\csc^2(\frac{\pi x}{2})\frac{\pi}{2}} = \frac{1}{-\frac{\pi}{2}} = \frac{-2}{\pi}$

Indeterminate Powers: If $\lim_{x\to a} f(x)^{g(x)}$ is of the form 0^0, ∞^0 or 1^∞, then the limit is an indeterminate power. To solve such a limit, take the natural logarithm, which converts the indeterminate power into an indeterminate product.

Example 3: Find the following limits, if they exist. If the limit does not exist, explain why.

(i) $\lim_{x\to\infty} x^{\frac{3}{x}}$ $= (\infty)^{\frac{3}{\infty}} = \infty^0$

$$y = x^{\frac{3}{x}}$$

$$\ln y = \ln x^{\frac{3}{x}}$$

$$\boxed{\ln y = \frac{3}{x}\ln x}$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{3}{x}\ln x = (0)(-\infty)$$

$$= \lim_{x\to\infty} \frac{3\ln x}{x} \quad \frac{\infty}{\infty}$$

$$\overset{L}{=} \lim_{x\to\infty} \frac{\frac{3}{x}}{1}$$

$$\lim_{x\to\infty} \ln y = 0 \rightarrow \boxed{y = e^0 = 1}$$

(ii) $\lim_{x\to\infty} \left(\frac{2x-3}{2x+5}\right)^{2x+1}$ $= (1)^\infty$

$$y = \left(\frac{2x-3}{2x+5}\right)^{2x+1}$$

$$\ln y = \ln\left(\frac{2x-3}{2x+5}\right)^{2x+1}$$

$$\ln y = (2x+1)\ln\left(\frac{2x-3}{2x+5}\right)$$

$$\frac{d}{dx}\frac{1}{2x+1} = \frac{d}{dx}(2x+1)^{-1} = -1(2x+1)^{-2}(2) = \frac{-2}{(2x+1)^2}$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} (2x+1)\ln\left(\frac{2x-3}{2x+5}\right) = (\infty)(\ln 1) = (\infty)(0)$$

$$= \lim_{x\to\infty} \frac{\ln\left(\frac{2x-3}{2x+5}\right)}{\frac{1}{2x+1}} \quad \frac{0}{0}$$

log properties

$$= \lim_{x\to\infty} \frac{\ln(2x-3) - \ln(2x+5)}{\frac{1}{2x+1}}$$

$$\overset{L}{=} \lim_{x\to\infty} \frac{\frac{2}{2x-3} - \frac{2}{2x+5}}{\frac{-2}{(2x+1)^2}}$$

$$= \lim_{x\to\infty} \frac{2(2x+5) - 2(2x-3)}{(2x-3)(2x+5)} \cdot \frac{(2x+1)^2}{-2}$$

$$4x + 10 - 4x + 6 = 16$$

$$= \lim_{x\to\infty} \frac{(\cancel{16})^{-8}(2x+1)^2}{(2x-3)(2x+5)(\cancel{-2})}$$

Foil + $4x^2$ leads

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leading powers

$$= \lim_{x\to\infty} \frac{(-8)(4x^2)}{4x^2} = -8$$

$$\lim_{x\to\infty} \ln y = -8, \quad y = e^{-8}$$

$$\boxed{\text{answer: } e^{-8}}$$

Indeterminate difference: If $\lim_{x\to a}(f(x) - g(x)) = \infty - \infty$, this limit is an indeterminate difference.

Example 4: Find $\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$ $= \lim_{x\to 1} \frac{x - 1 - \ln x}{(\ln x)(x-1)} \quad \frac{0}{0}$

$$\overset{L}{=} \lim_{x\to 1} \frac{1 - \frac{1}{x}}{\frac{1}{x}(x-1) + \ln x} \quad \frac{0}{0}$$

$$1 - \frac{1}{x}$$

$$= \lim_{x\to 1} \frac{\frac{1}{x^2}}{\frac{1}{x^2} + \frac{1}{x}} = \boxed{\frac{1}{2}}$$