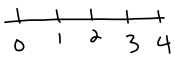
Section 6.3: The Definite Integral

Recall If $f(x) \geq 0$ on the interval $[a, b]$, the area under the curve of $f(x)$, above the x-axis, from $x = a$ to $x = b$ is $\approx \sum_{i=1}^{n} f(x_i^*)\Delta x_i$. We call this sum a **Riemann Sum.**

↑ sum of approximating rectangles

EXAMPLE 1: If $f(x) = x + x^2$, the interval $[0, 4]$, partition points $P = \{0, 1, 2, 3, 4\}$, and $x_i^* =$ left endpoint. Find the Riemann Sum.



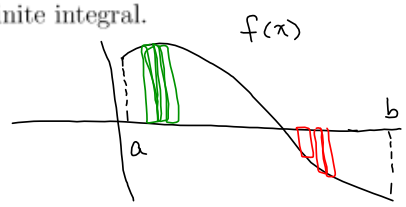
sub-int	x_i^* = left	Δx_i
$[0,1]$	0	1
$[1,2]$	1	1
$[2,3]$	2	1
$[3,4]$	3	1

$$L_4 = \sum_{i=1}^{4} f(x_i^*)\Delta x_i = \big(f(0) + f(1) + f(2) + f(3)\big)(1)$$

$$= 0 + 2 + 6 + 12$$

$$= \boxed{20}$$

Recall If $f(x) \geq 0$ on the interval $[a, b]$, then the true area under the graph of $f(x)$ from $x = a$ to $x = b$ is $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$ where $\Delta x_i = \dfrac{b-a}{n}$ and x_i^* is any point on the ith subinterval. We would like to define the limit of a Riemann Sum irregardless of whether the function is positive. To that end, we will define the definite integral.



Definition The Definite Integral of $f(x)$ from $x = a$ to $x = b$ is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

$\int$ ← integral sign = anti derivative

$\int_a^b$ ← definite integral

where $\Delta x_i = \dfrac{b-a}{n}$ and x_i^* is any point on the ith subinterval. In the event $f(x)$ is positive on the interval $[a, b]$, then the definite integral is the same as the area bounded by $f(x)$, the x-axis, $x = a$ and $x = b$. If $f(x)$ is not always positive on the interval $[a, b]$, then the definite integral is the net area.

Fact Any Riemann sum can approximate a definite integral. Specifically, the **Midpoint Rule** can be used to approximate the definite integral.

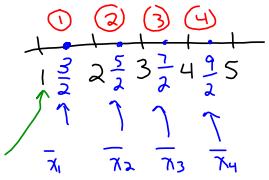
Midpoint Rule: $\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\overline{x_i})\Delta x$, where $\Delta x = \dfrac{b-a}{n}$ and $\overline{x_i}$ is the midpoint of the ith subinterval.

EXAMPLE 2: Use the Midpoint Rule with $n = 4$ to approximate $\int_1^5 \sqrt{x^2+1}\,dx$.

$I = [1, 5]$

$\overline{x_i}$ = midpoints

$\Delta x = \dfrac{b-a}{n} = \dfrac{5-1}{4}$



$\sum_{i=1}^{4} f(\overline{x_i})\,\Delta x_i$

$\Delta x = 1$

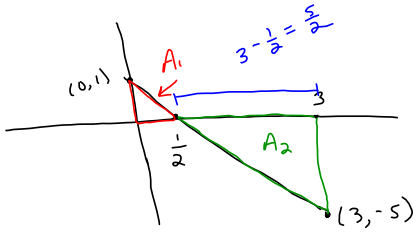
$$f\left(\tfrac{3}{2}\right)(1) + f\left(\tfrac{5}{2}\right)(1) + f\left(\tfrac{7}{2}\right)(1) + f\left(\tfrac{9}{2}\right)(1)$$

$$\sqrt{\tfrac{9}{4}+1} + \sqrt{\tfrac{25}{4}+1} + \sqrt{\tfrac{49}{4}+1} + \sqrt{\tfrac{81}{4}+1}$$

EXAMPLE 3: Use Geometry to evaluate the following definite integrals:

(i) $\int_0^3 (1-2x)\,dx$

Recall: $\int_a^b f(x)\,dx$ can be interpreted as area.



$\int_0^3 (1-2x)\,dx$ = net area

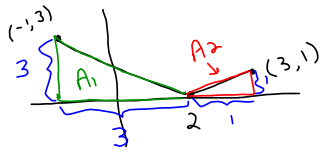
$= A_1 - A_2$

$= \frac{1}{2}\left(\frac{1}{2}\right)(1) - \frac{1}{2}\left(\frac{5}{2}\right)(5)$

$= \frac{1}{4} - \frac{25}{4}$

$= \boxed{-6}$

(ii) $\int_{-1}^3 |x-2|\,dx$



$\int_{-1}^3 |x-2|\,dx = A_1 + A_2$

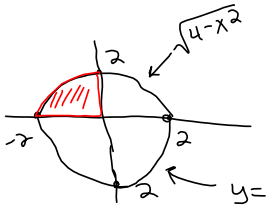
$= \frac{1}{2}(3)(3) + \frac{1}{2}(1)(1)$

$= \frac{9}{2} + \frac{1}{2} = \boxed{5}$

(iii) $\int_{-2}^0 \sqrt{4-x^2}\,dx$

$x^2 + y^2 = 4$

$y = \pm\sqrt{4-x^2}$



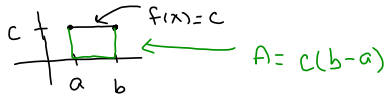
$\sqrt{r^2 - x^2}$

$\int_{-2}^0 \sqrt{4-x^2}\,dx = \frac{1}{4}\pi(2)^2$

$= \boxed{\pi}$

Theorem

(a) $\int_a^b c\,dx = c(b-a)$



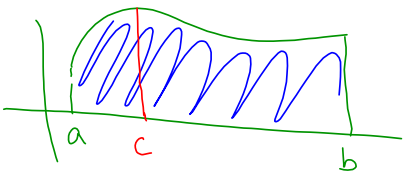
(b) $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

(c) $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

(d) $\int_a^a f(x)\,dx = 0$ ← zero area

(e) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ ←

(f) $\int_a^b dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$



(g) If $m \le f(x) \le M$ for all x in the interval $[a, b]$, then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$.

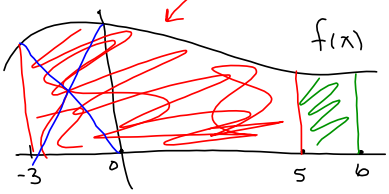
EXAMPLE 4: Find $\int_1^1 \sqrt{x^5 + x^2 + 1}\,dx$ = 0

EXAMPLE 5: If $\int_1^3 f(x)\,dx = 4$ and $\int_1^3 g(x)\,dx = -3$, find $\int_1^3 (f(x) + 2g(x))\,dx$.

$\int_1^3 f(x)\,dx + 2\int_1^3 g(x)\,dx$ (4, -3)

$4 - 6 = \boxed{-2}$

EXAMPLE 6: Write $\int_{-3}^{5} f(x)\,dx - \int_{-3}^{0} f(x)\,dx + \int_{5}^{6} f(x)\,dx$ as a single integral.



$\int_0^6 f(x)\,dx$

$\int_{-3}^{5} f(x)\,dx + \int_{0}^{-3} f(x)\,dx + \int_{5}^{6} f(x)\,dx$

$\int_0^5 f(x)\,dx + \int_5^6 f(x)\,dx = \int_0^6 f(x)\,dx$

EXAMPLE 7: Find an upper and lower bound on $\int_0^2 \sqrt{x^3+1}\,dx$.

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EXAMPLE 8: Express the following limits as a definite integral:

(i) $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{1}{1+(i/n)^2}$

$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$

$a = ?$
$b = ?$
$f(x) = ?$

$\Delta x = \frac{b-a}{n} = \frac{1}{n}$ $\quad b-a = 1$

$x_i^* = a + \left(\frac{b-a}{n}\right) i = a + (\Delta x)\,i$

"What is being plugged in for x?"

$\int_0^1 \frac{1}{1+x^2}\,dx$

$x_i^* = \frac{i}{n}$ $\quad a = 0$ $\quad b = 1$

(ii) $\lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n} \left(3(1+\frac{2i}{n})^5 - 6\right)$

$\Delta x = \frac{2}{n}$ $\quad b-a = 2$

$x_i^* = a + \frac{b-a}{n}\,i = 1 + \frac{2}{n}\,i$

$a = 1$
$b = 3$

$\int_1^3 (3x^5 - 6)\,dx$

both work

$\int_0^2 \left(3(1+x)^5 - 6\right)$