

2. Find the sum: $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!}$.

- (a) None of these
- (b) -1
- (c) 0
- (d) $\frac{1}{1 + \pi^2}$
- (e) $e^{-\pi^2}$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\Rightarrow \cos \pi = -1$$

Find the sum

$$(a) \sum_{n=1}^{\infty} \frac{4^n}{n!}$$

$$\text{Know: } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^4 = \sum_{n=0}^{\infty} \frac{4^n}{n!}$$

Fact: $\sum_{n=0}^{\infty} a_n = a_0 + \sum_{n=1}^{\infty} a_n$

$$(b) \sum_{n=2}^{\infty} \frac{4^n}{n!}$$

$$e^4 = 1 + \sum_{n=1}^{\infty} \frac{4^n}{n!}$$

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + \sum_{n=2}^{\infty} a_n$$

$$e^4 = 1 + 4 + \sum_{n=2}^{\infty} \frac{4^n}{n!}$$

$$e^4 - 5 = \sum_{n=2}^{\infty} \frac{4^n}{n!}$$

3. Which statement is true regarding the convergence or divergence of $\sum_{n=1}^{\infty} \frac{1 + \sin^2 n}{n^2}$?

- (a) The series converges by the Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{2}{n^2}$.
- (b) The series diverges by the Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{n}$.
- (c) The series converges by the Comparison Test with $\sum_{n=1}^{\infty} \frac{2}{n^2}$.
- (d) None of the other statements is true.
- (e) The series diverges by the Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{n}$.

5. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges to s . Based on the Alternating Series Estimation Theorem, which statement is true?

- (a) $|s - s_7| < \frac{1}{49}$
- (b) $|s - s_7| < \frac{1}{7}$
- (c) None of these
- (d) $|s - s_7| < \frac{1}{8}$
- (e) $|s - s_7| < \frac{1}{64}$

If $\sum_{n=1}^{\infty} (-1)^n a_n$ converges,

$$\text{then } |R_n| < a_{n+1}$$

$$|s - S_n| < a_{n+1}, a_n = \frac{1}{n^2}$$

here, $n=7$

$$|s - S_7| < a_8 = \frac{1}{64}$$

6. Given the power series for $\frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$ for $-1 < x < 1$, which of the following is a power series for

$$f(x) = \frac{1}{(1-x)^2}?$$

- (a) $\sum_{n=0}^{\infty} x^{2n}$
- (b) $\sum_{n=1}^{\infty} nx^{n-1}$
- (c) $\sum_{n=1}^{\infty} (-1)nx^{n-1}$
- (d) $\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$
- (e) None of these

$$\int \frac{1}{(1-x)^2} dx = \frac{1}{1-x}$$

$u = 1-x$
 $du = -dx$

$$\begin{aligned} &= \frac{d}{dx} \sum_{n=0}^{\infty} x^n \\ &= \boxed{\sum_{n=1}^{\infty} n x^{n-1}} = \sum_{n=0}^{\infty} (n+1)x^n \end{aligned}$$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=0}^{\infty} a_{n+1}$$

11. Which statement is true regarding the series $\sum_{n=1}^{\infty} (-1)^n \frac{4}{\sqrt{n^2 + 4} + n}$: is convergent by AST

- (a) None of the other statements is true.
- (b) The series is absolutely convergent by the Test for Divergence.
- (c) The series is divergent.
- (d) The series is convergent, but not absolutely convergent.
- (e) The series is absolutely convergent by the Ratio Test.

"Absolute convergence Test"

If $\sum |a_n|$ converges, then $\sum a_n$ is absolutely convergent.

$$\sum \left| (-1)^n \frac{4}{\sqrt{n^2 + 4} + n} \right|$$

$$= \sum \frac{4}{\sqrt{n^2 + 4} + n} \text{ behave like } \sum \frac{4}{2n} \text{ divergent P-series}$$

(in FR, you would need to do LCT)

12. Given the Maclaurin Series for a function f is defined by $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^n}{n 2^n}$, what is $f^{(10)}(0)$ (the tenth derivative of f at $x = 0$)?

- (a) $-\frac{1}{10 \cdot 2^{10}}$
- (b) None of these
- (c) $-\frac{9!}{2^{10}}$
- (d) $\frac{1}{10 \cdot 2^{10}}$
- (e) $\frac{9!}{2^{10}}$

maclaurin series: $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$
coefficient of the maclaurin series is $c_n = \frac{f^{(n)}(0)}{n!}$

coefficient of the series they gave me is $c_n = \frac{(-1)^{n+1}}{n 2^n}$

$$\frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n+1}}{n 2^n}$$

$$\frac{f^{(10)}(0)}{10!} = \frac{(-1)^{11}}{10 \cdot 2^{10}} \rightarrow f^{(10)}(0) = -\frac{1}{10 \cdot 2^{10}} 10!$$

$$= -\frac{1}{10 \cdot 2^{10}} 10 \cdot 9!$$

$$= -\frac{9!}{2^{10}}$$

13. For which series below is the Ratio Test inconclusive?

- (a) $\sum_{n=0}^{\infty} \frac{1}{2^n}$
- (b) $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n!}$
- (c) None of these
- (d) $\sum_{n=0}^{\infty} \frac{3^n}{5^{n-1}}$
- (e) $\sum_{n=0}^{\infty} \frac{(-1)^n}{4n^2 + 3n + 3}$

14. Which of the following is the 3rd degree Taylor Polynomial for $f(x) = x + \sin(2x)$ centered at $a = 0$?

- (a) $T_3(x) = 3x - \frac{4}{3}x^3$
- (b) None of these
- (c) $T_3(x) = 3x - 8x^3$
- (d) $T_3(x) = 2x - \frac{1}{3}x^3$
- (e) $T_3(x) = 2x - 2x^3$

$$\begin{aligned} f(0) &= 0 \\ f'(x) &= 1 + 2 \cos(2x) & f'(0) &= 3 \\ f''(x) &= -4 \sin(2x) & f''(0) &= 0 \\ f'''(x) &= -8 \cos(2x) & f'''(0) &= -8 \end{aligned}$$

$$\begin{aligned} T_3(x) &= \frac{f(0)}{0!} x^0 + \frac{f'(0)}{1!} x^1 + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 \\ &= 3x - \frac{8}{6} x^3 \end{aligned}$$

15. What is the radius of convergence of $\sum_{n=0}^{\infty} \frac{(2012)^n x^n}{n!}$?

- (a) ∞
- (b) None of these
- (c) $\frac{1}{2012}$
- (d) 0
- (e) 2012

$$\lim_{n \rightarrow \infty} \left| \frac{(2012)^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{(2012)^n x^n} \right|$$

$R = \infty$

$$= \lim_{n \rightarrow \infty} \left| \frac{(2012) x}{n+1} \right| = 0 < 1$$

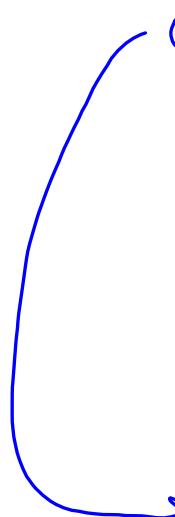
16. (5 points) Find the Taylor Series for $f(x) = e^{-x}$ centered at $a = 3$.

$f(x)$

$$e^{-x} = \sum_{n=0}^{\infty} \frac{f(3)}{n!} (x-3)^n$$

$$f'(x) = (-1) e^{-x}$$

$$f'(3) = (-1)^3 e^{-3}$$



$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n e^{-3}}{n!} (x-3)^n$$

$$f(x) = e^{-x}$$

$$f'(x) = -e^{-x}$$

$$f''(x) = e^{-x}$$

$$f'''(x) = -e^{-x}$$

17.

- (a) (6 points) Find a power series representation for $f(x) = \frac{1}{2+4x}$ about $a = 0$.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad |x| < 1$$

$$\frac{1}{2(1+2x)} = \frac{1}{2} \sum_{n=0}^{\infty} (-2x)^n \quad |-2x| < 1$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n 2^n x^n \quad |x| < \frac{1}{2}$$

$$\frac{1}{2+4x} = \boxed{\sum_{n=0}^{\infty} (-1)^n 2^{n-1} x^n}$$

- (b) (2 points) What is the radius of convergence of the series in part (a)?

$$R = \frac{1}{2}$$

- (c) (4 points) Use your series in part (a) to find the Maclaurin series for $\ln(2+4x)$

$$\int \frac{d}{dx} \ln(2+4x) dx = \frac{4}{2+4x} = 4 \left(\frac{1}{2+4x} \right)$$

$$= 4 \sum_{n=0}^{\infty} (-1)^n 2^{n-1} x^n dx$$

$$\ln(2+4x) = C + 4 \sum_{n=0}^{\infty} (-1)^n 2^{n-1} \frac{x^n}{n+1}$$

$$x=0 \quad \ln(2)=C$$

18. (8 points) Find the radius and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n(x-2)^n}{\sqrt[3]{n+2}}$. Be sure to test the endpoints for convergence.

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^n (x-2)^n}{\sqrt[3]{n+3}} \cdot \frac{\sqrt[3]{n+2}}{(-1)^n (x-2)^n} \right| = |x-2| < \frac{1}{R} \quad R=1$$

$$-1 < x-2 < 1$$

Test endpoints

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{\sqrt[3]{n+2}}$$

$$x=3 : \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n+2}}$$

AST converges since

$$\frac{1}{\sqrt[3]{n+2}} \text{ decreases to } 0$$

$$x=1 : \sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n}{\sqrt[3]{n+2}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n+2}}$$

$$\text{CT} \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n+2}} < \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$$

$$\text{LCT} \quad \lim_{n \rightarrow \infty} \frac{\frac{1}{\sqrt[3]{n+2}}}{\frac{1}{\sqrt[3]{n}}} =$$

$$\text{CT} \quad \text{FAILS}$$

$$\lim_{n \rightarrow \infty} \frac{\sqrt[3]{n}}{\sqrt[3]{n+2}} = 1 > 0$$

$$I = [1, 3]$$

Both diverge since

$\sum \frac{1}{\sqrt[3]{n}}$ diverges by p-series
 $p = \frac{1}{3}$

13. (12 pts) Express $f(x) = \ln(5 - x^3)$ as a power series about $x = 0$ and identify the radius of convergence.

$$\frac{d}{dx} \ln(5 - x^3) = \frac{-3x^2}{5 - x^3}$$

$$= \frac{-3x^2}{5(1 - \frac{x^3}{5})}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$= -\frac{3x^2}{5} \sum_{n=0}^{\infty} \left(\frac{x^3}{5}\right)^n$$

$$\left| \frac{x^3}{5} \right| < 1$$

$$= -\frac{3x^2}{5} \sum_{n=0}^{\infty} \frac{x^{3n}}{5^n}$$

$$\begin{aligned} |x| &< \sqrt[3]{5} \\ R &= \sqrt[3]{5} \end{aligned}$$

$$\int \frac{d}{dx} \ln(5 - x^3) dx = \int -3 \sum_{n=0}^{\infty} \frac{x^{3n+2}}{5^{n+1}} dx$$

$$\ln(5 - x^3) = C - 3 \sum_{n=0}^{\infty} \frac{x^{3n+3}}{(3n+3)(5^{n+1})}$$

$$\text{Find } C: \quad \ln 5 = C$$

$x=0$

4. (4 pts) Using the Alternating Series Estimation Theorem, how many terms of the series do we need to add in order to find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ with error less than $\frac{1}{60}$?

- (a) $n = 3$
- (b) $n = 4$
- (c) $n = 5$
- (d) $n = 7$
- (e) $n = 6$

$$\text{AS ET : } |R_n| \leq \frac{1}{(n+1)^2} < \frac{1}{60}$$

$$60 < (n+1)^2$$

$$|R_n| \leq a_{n+1}$$

$$a_n = \frac{1}{n^2}$$

$n = 6$? $60 < 49$ NOPE

$n = 7$? $60 < 64$ YEP!

5. (4 pts) Which of the following series converge absolutely?

- (a) $\sum_{n=1}^{\infty} (-1)^n$ → diverges by TD
- (b) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n\sqrt{n}}$ → $\sum \left| \frac{\cos(n\pi)}{n\sqrt{n}} \right| = \sum \frac{1}{n\sqrt{n}}$ converges!
- (c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ → $\sum \frac{1}{\sqrt{n}}$ diverges → $\sum \frac{(-1)^n}{\sqrt{n}}$ does not CA
- (d) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

- (e) All of the above series are absolutely convergent.

$$\sum_{n=2}^{\infty} \frac{1}{\ln n} > \sum \frac{1}{n} \text{ diverges!}$$

7. (4 pts) The Interval of Convergence of the series $\sum_{n=0}^{\infty} \frac{(x+1)^n(2n+1)!}{10^n n!}$ is:

- (a) $I = (-11, 9)$
- (b) $I = \left(-\frac{1}{10}, \frac{1}{10}\right)$
- (c) $I = [-11, 9)$
- (d) $I = (-\infty, \infty)$
- (e) $I = \{-1\}$

$$I = \{-1\} \quad \stackrel{R \neq 0}{=} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$$

and $R = 0$ unless $x = -1$,

$$\sum_{n=0}^{\infty} \frac{(x+1)^n n! 10^n}{(2n+1)!} \rightarrow R = \infty$$

$I = (-\infty, \infty) \cap \mathbb{R}$

9. (4 pts) $\arctan(x^3) =$

- (a) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+4}}{2n+1}$
- (b) $\sum_{n=0}^{\infty} \frac{x^{6n+3}}{2n+1}$
- (c) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{2n+1}$
- (d) $\sum_{n=0}^{\infty} \frac{x^{6n+1}}{6n+1}$
- (e) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{6n+1}$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$\arctan(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n+1}}{2n+1}$$

12. (i) (4 pts) Find a Maclaurin Series representation for $f(x) = \sin\left(\frac{x^2}{3}\right)$.

$$\sin\left(\frac{x^2}{3}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{x^2}{3}\right)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)! 3^{2n+1}}$$

(ii) (6 pts) Using the result in part (i), write $\int_0^1 \sin\left(\frac{x^2}{3}\right) dx$ as an infinite series.

$$= \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)! 3^{2n+1}} dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)! 3^{2n+1} (4n+3)} \Big|_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! 3^{2n+1} (4n+3)}$$

6. Consider the series below. Which statement is true regarding the absolute convergence of each series?

$$(I) \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2 + 3} \quad (II) \sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{4^n + 1}$$

- (a) I converges absolutely, II converges but not absolutely
- (b) Both I and II converge but not absolutely
- (c) Both I and II converge absolutely
- (d) I diverges, II converges absolutely
- (e) I converges but not absolutely, II converges absolutely

$$\text{I. } \sum_{n=1}^{\infty} \left| \frac{(-1)^n n}{n^2 + 3} \right| = \sum_{n=1}^{\infty} \frac{n}{n^2 + 3} \sim \sum \frac{1}{n}$$

$$\text{II. } \sum_{n=1}^{\infty} \left| \frac{(-1)^n 3^n}{4^n + 1} \right| = \sum_{n=1}^{\infty} \frac{3^n}{4^n + 1} \leq \sum_{n=1}^{\infty} \frac{3^n}{4^n} \quad r = \frac{3}{4} < 1$$

8. Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1}}{4^{2n} (2n)!}$

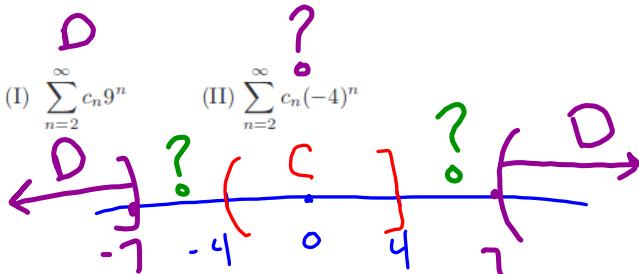
$$= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} \cdot 3}{4^{2n} (2n)!}$$

- (a) $3 \cos\left(\frac{3}{4}\right)$
- (b) $3 \sin\left(\frac{3}{4}\right)$
- (c) $\cos\left(\frac{3}{4}\right)$
- (d) $\sin\left(\frac{3}{4}\right)$
- (e) $3 \sin\left(-\frac{3}{4}\right)$

$$= 3 \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{3}{4}\right)^{2n}}{(2n)!} = 3 \cos \frac{3}{4}$$

9. The series $\sum_{n=2}^{\infty} c_n x^n$ converges when $x = 4$ and diverges when $x = -7$. What can be said about the convergence of the following series?

- (a) Both I and II are inconclusive
- (b) I diverges, II converges
- (c) I diverges, II is inconclusive
- (d) Both I and II converge
- (e) I is inconclusive, II converges



23. (10 pts) Express $f(x) = \frac{4x}{(x^2 + 4)^2}$ as a power series centered at zero. Include the radius of convergence in your answer.

$$f(x) = \frac{4x}{(x+4)^2} = 4x \left(\frac{1}{(x+4)^2} \right)$$

$$\begin{aligned} \textcircled{1} \quad & \frac{d}{dx} \int \frac{1}{(x+4)^2} dx = -\frac{1}{x+4} \\ &= -\frac{1}{4(1+\frac{x}{4})} \\ &= -\frac{1}{4} \sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n \\ &= -\frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{4^n} \\ &\frac{1}{(x+4)^2} = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{4^{n+1}} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n n x^n}{4^{n+1}} \\ &\frac{1}{(x+4)^2} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1) x^{n+2}}{4^{n+2}} \\ \textcircled{2} \quad & \text{multiply by } 4x \quad \frac{4x}{(x+4)^2} = 4x \sum_{n=0}^{\infty} \frac{(-1)(n+1)x^n}{4^{n+2}} \\ &= \sum_{n=0}^{\infty} \frac{(-1)(n+1)x^n}{4^{n+1}} \end{aligned}$$

7. Given the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n (x-3)^n}{2^n \cdot n^4}$ is 2, what is the interval of convergence?

- (a) $(-2, 2)$
- (b) $(-2, 2]$
- (c) None of these
- (d) $[1, 5]$
- (e) $(1, 5)$



$$x=5 \quad \sum \frac{(-1)^n 2^n}{2^n n^4} \quad \text{converges AST}$$

$$x=1 \quad [1, 5] \quad \sum \frac{(-1)^n (-2)^n}{2^n n^4} = \sum \frac{2^n}{2^n n^4} = \sum \frac{1}{n^4}$$

8. Given the power series for $\frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$ for $-1 < x < 1$, which of the following is a power series for

$$f(x) = \frac{1}{(1-x)^2} ?$$

(a) $\sum_{n=1}^{\infty} (-1)nx^{n-1}$

(b) $\sum_{n=0}^{\infty} x^{2n}$

(c) $\sum_{n=1}^{\infty} nx^{n-1}$

(d) $\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$

(e) None of these

$$\begin{aligned} f(x) &= \frac{1}{(1-x)^2} \\ \frac{d}{dx} \int \frac{1}{(1-x)^2} dx &= \frac{1}{1-x} = \frac{d}{dx} \sum_{n=0}^{\infty} x^n \\ &= \sum_{n=1}^{\infty} n x^{n-1} \end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{6n+2}$$

T.O. $\lim_{n \rightarrow \infty} \frac{(-1)^n}{6n+2} \neq 0$