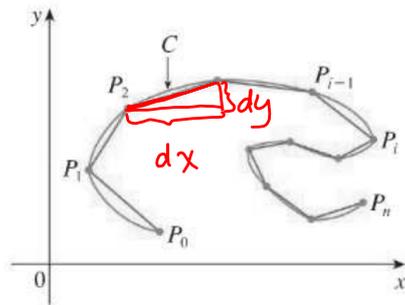


Calculus with Parametric Curves

Arclength If $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, then the length of the curve from $t = \alpha$ to $t = \beta$ is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$



$$L = \sqrt{(dx)^2 + (dy)^2}$$

1. Find the length of the curve $x = 3t - t^3$, $y = 3t^2$, $0 \leq t \leq 2$.

$$L = \int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$= \int_0^2 \sqrt{(3-3t^2)^2 + (6t)^2} dt$$

~~$$= \int_0^2 (3-3t^2 + 6t) dt$$~~

$$\int_0^2 \sqrt{9-18t^2+9t^4+36t^2} dt$$

$$\int_0^2 \sqrt{9t^4+18t^2+9} dt$$

$$\int_0^2 \sqrt{(3t^2+3)^2} dt$$

$$\int_0^2 (3t^2+3) dt = (t^3+3t) \Big|_0^2$$

$$= \boxed{14}$$

2. Find the length of the curve $x = 2t^2 + \frac{1}{t}$, $y = 8\sqrt{t}$, $1 \leq t \leq 3$

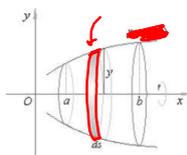
$$\begin{aligned}
 L &= \int_1^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
 &= \int_1^3 \sqrt{\left(4t - \frac{1}{t^2}\right)^2 + \left(\frac{8}{2\sqrt{t}}\right)^2} dt \\
 &= \int_1^3 \sqrt{16t^2 - \frac{8}{t} + \frac{1}{t^4} + \frac{16}{t}} dt \\
 &= \int_1^3 \sqrt{16t^2 + \frac{8}{t} + \frac{1}{t^4}} dt \\
 &= \int_1^3 \sqrt{\left(4t + \frac{1}{t^2}\right)^2} dt \\
 &= \int_1^3 \left(4t + \frac{1}{t^2}\right) dt = \left(2t^2 - \frac{1}{t}\right) \Big|_1^3 \\
 &= 18 - \frac{1}{3} - 2 + 1
 \end{aligned}$$

$$\begin{aligned}
 &\left(4t - \frac{1}{t^2}\right)^2 \\
 &= 16t^2 - 2 \cdot 4t \cdot \frac{1}{t^2} + \frac{1}{t^4} \\
 &= 16t^2 - \frac{8}{t} + \frac{1}{t^4}
 \end{aligned}$$

Surface Area

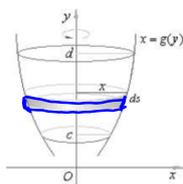
• If the curve $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, is revolved around the x axis, then the resulting surface area is given by

$$SA = 2\pi \int_{\alpha}^{\beta} g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$



• If the curve $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, is revolved around the y axis, then the resulting surface area is given by

$$SA = 2\pi \int_{\alpha}^{\beta} f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$



3. Find the surface area obtained by revolving the curve $x = 3t - t^3$, $y = 3t^2$, $0 \leq t \leq 1$ about the x -axis.

$$\begin{aligned}
 SA &= \int_0^1 2\pi \cdot 3t^2 \sqrt{(3-3t^2)^2 + (6t)^2} dt \\
 &= 6\pi \int_0^1 t^2 (3t^2 + 3) dt \quad \leftarrow \text{From problem 1} \\
 &= 6\pi \int_0^1 (3t^4 + 3t^2) dt = 6\pi \left(\frac{3t^5}{5} + t^3 \right) \Big|_0^1 \\
 &= 6\pi \left(\frac{3}{5} + 1 \right) = 6\pi \left(\frac{8}{5} \right)
 \end{aligned}$$

4. Find the surface area obtained by revolving the curve $x = e^t - t$, $y = 4e^{t/2}$, $0 \leq t \leq 1$ about the y -axis.

$$\begin{aligned}
 &\int_0^1 2\pi (e^t - t) \sqrt{(e^t - 1)^2 + \left[4\left(\frac{1}{2} e^{t/2}\right) \right]^2} dt \\
 &= 2\pi \int_0^1 (e^t - t) \sqrt{e^{2t} - 2e^t + 1 + 4e^t} dt \\
 &= 2\pi \int_0^1 (e^t - t) \sqrt{e^{2t} + 2e^t + 1} dt \\
 &= 2\pi \int_0^1 (e^t - t) \sqrt{(e^t + 1)^2} dt \\
 &= 2\pi \int_0^1 (e^t - t)(e^t + 1) dt \\
 &= 2\pi \int_0^1 (e^{2t} + e^t - \underline{te^t} - t) dt \\
 &= 2\pi \left(\frac{1}{2} e^{2t} + \underline{e^t} - \left(te^t - \underline{e^t} \right) - \frac{t^2}{2} \right) \Big|_0^1 \\
 &= 2\pi \left(\frac{1}{2} e^{2t} + 2e^t - te^t - \frac{t^2}{2} \right) \Big|_0^1 \\
 &= 2\pi \left(\frac{1}{2} e^2 + 2e - e - \frac{1}{2} - \left(\frac{1}{2} + 2 \right) \right)
 \end{aligned}$$

u	dv
$t \oplus e^t$	e^t
$1 \ominus e^t$	e^t
0	e^t