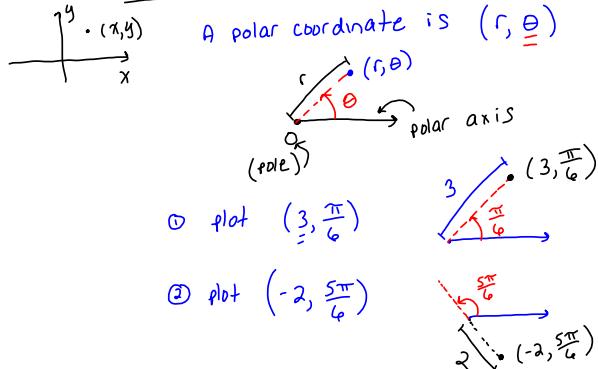


NO week in review tomorrow  
 Final exam review: Wednesday, May 1,  
 1-3 PM HELD 100

Polar coordinate

First a cartesian coordinate is  $(x, y)$

A polar coordinate is  $(r, \theta)$

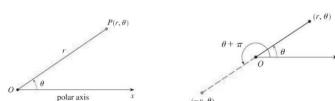


### Section 10.3 Polar Coordinates

A coordinate system represents a point in the plane by an ordered pair of numbers called coordinates. So far we have been using cartesian coordinates, which are directed distances from two perpendicular axes. In this section we describe a coordinate system called **the polar coordinate system** which is more convenient for many purposes.

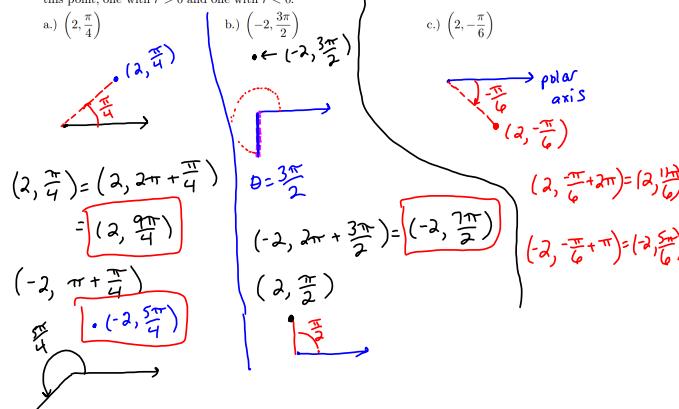
We choose a point in the plane that is called the **pole** (or origin) and labeled  $O$ . Then we draw a ray starting at  $O$  called the **polar axis**. This axis is usually drawn horizontally to the right and corresponds to the positive  $x$ -axis in cartesian coordinates.

If  $P$  is any other point in the plane, let  $r$  be the distance from  $O$  to  $P$  and let  $\theta$  be the angle between the polar axis and the line  $OP$ . Then the point  $P$  is represented by the ordered pair  $(r, \theta)$ , and  $r, \theta$  are called the **polar coordinates** of  $P$ .

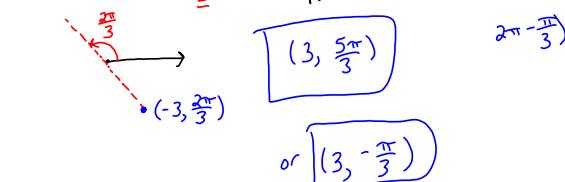


Note: We use the convention that an angle is positive if measured in the counterclockwise direction and negative in the clockwise direction. The meaning of polar coordinates  $(r, \theta)$  can be extended to the case in which  $r$  is negative by taking the convention that  $(-r, \theta) = (r, \theta + \pi)$ . Also note that  $(r, \theta) = (r, 2\pi + \theta)$ , so there are infinitely many ways to express a point in polar coordinates.

1. Plot the points whose polar coordinates are given. Then find two other pairs of polar coordinates at this point, one with  $r > 0$  and one with  $r < 0$ .



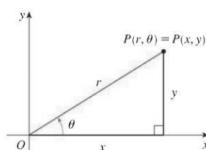
plot  $(-3, \frac{2\pi}{3})$ , give one more polar point where  $r > 0$



plot  $(r, \theta) = (-5, 0)$  ← polar point!

$$(-5, 0) = (5, \pi)$$

To convert from polar coordinates  $((r, \theta))$  to cartesian coordinates  $((x, y))$ , or vice versa:



$$\cos \theta = \frac{A}{H} = \frac{x}{r} \rightarrow x = r \cos \theta$$

$$\sin \theta = \frac{O}{H} = \frac{y}{r} \rightarrow y = r \sin \theta$$

$$\tan \theta = \frac{O}{A} = \frac{y}{x} \rightarrow \theta = \arctan \frac{y}{x}$$

a.)  $x = r \cos(\theta)$ ,  $y = r \sin(\theta)$

b.)  $\tan(\theta) = \frac{y}{x}$ , thus  $\theta = \arctan\left(\frac{y}{x}\right)$

c.)  $x^2 + y^2 = r^2$

2. Find the cartesian coordinates of the polar point  $(2, \frac{2\pi}{3})$ .  $= (r, \theta)$

here,  $r = 2$

$\theta = \frac{2\pi}{3}$

$$x = r \cos \theta = 2 \cos \frac{2\pi}{3} = 2\left(-\frac{1}{2}\right) = -1$$

$$y = r \sin \theta = 2 \sin \frac{2\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

cartesian coordinate  $= (-1, \sqrt{3})$

3. Find a cartesian equation for the curve described by the given polar equation and identify it.

a.)  $r = 2 \sin \theta$

$r = 2 \sin \theta$

know:  $y = r \sin \theta$

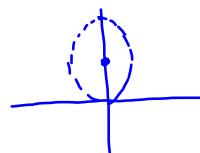
$$\begin{matrix} r^2 &= 2r \sin \theta \\ x^2 + y^2 &= 2y \end{matrix}$$

$$x^2 + y^2 = 2y$$

$$x^2 + y^2 - 2y + 1 = 0 + 1$$

$$x^2 + (y-1)^2 = 1$$

circle



b.)  $r = 9 \cos \theta$

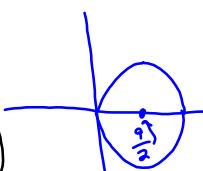
$$r^2 = 9r \cos \theta$$

$$x^2 + y^2 = 9x$$

$$x^2 - 9x + \frac{81}{4} + y^2 = 0 + \frac{81}{4}$$

$$\left(x - \frac{9}{2}\right)^2 + y^2 = \frac{81}{4}$$

circle



c.)  $r = 5 \sec \theta$

$$r = \frac{5}{\cos \theta}$$

$$r \cos \theta = 5$$

$$x = 5$$

vertical line!



4. Find polar coordinates,  $(r, \theta)$ , of the rectangular point, where  $r > 0$ .

a.)  $(3, \sqrt{3}) = (x, y)$

$$x=3, y=\sqrt{3}$$

Find  $(r, \theta)$

$$r^2 = x^2 + y^2 = 9 + 3 = 12$$

$$r = \sqrt{12} = 2\sqrt{3}$$

$$\tan \theta = \frac{y}{x} = \frac{\sqrt{3}}{3} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{1}{3}$$

$$\tan \theta = \arctan\left(\frac{1}{3}\right)$$

$$\theta = \frac{\pi}{6}$$

$$\boxed{(2\sqrt{3}, \frac{\pi}{6})}$$

b.)  $(3, 4) = (x, y)$

$$\begin{aligned} x &= 3 \\ y &= 4 \\ r^2 &= x^2 + y^2 = 9 + 16 \\ &= 25 \end{aligned}$$

$$r = 5$$

$$\theta = \arctan \frac{4}{3}$$

$$\boxed{(5, \arctan \frac{4}{3})}$$

c.)  $(-2, 2) = (x, y)$

$$\begin{aligned} x &= -2, y = 2 \\ r^2 &= x^2 + y^2 = 4 + 4 = 8 \end{aligned}$$

$$\boxed{r = \sqrt{8} = 2\sqrt{2}}$$

$$\tan \theta = \frac{y}{x} = \frac{2}{-2} = -1$$

$$\boxed{\theta = \frac{3\pi}{4} \text{ or } \theta = -\frac{\pi}{4}}$$

since original point is in QII

$$\boxed{(2\sqrt{2}, \frac{3\pi}{4})}$$

5. Find a polar equation for the curve described by the given Cartesian equation.

a.)  $x = 6$

$$\boxed{r \cos \theta = 6}$$

b.)  $y + x = 1$

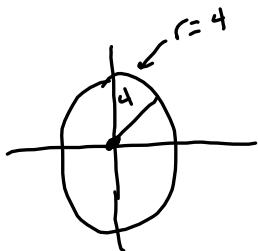
$$\boxed{r \sin \theta + r \cos \theta = 1}$$

c.)  $x^2 = 4y$

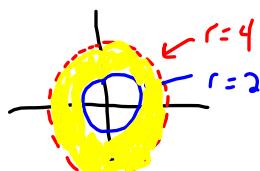
$$\boxed{r^2 \cos^2 \theta = 4r \sin \theta}$$

6. Sketch the region in the plane consisting of points whose polar coordinates are given.

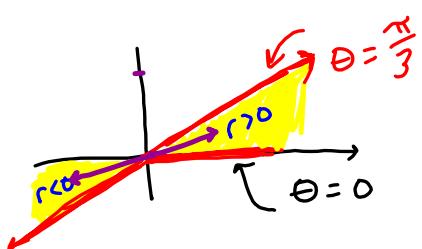
a.)  $r = 4$



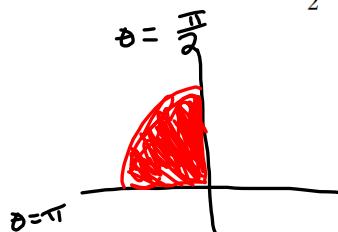
b.)  $2 \leq r < 4$



c.)  $0 \leq \theta \leq \frac{\pi}{3}$

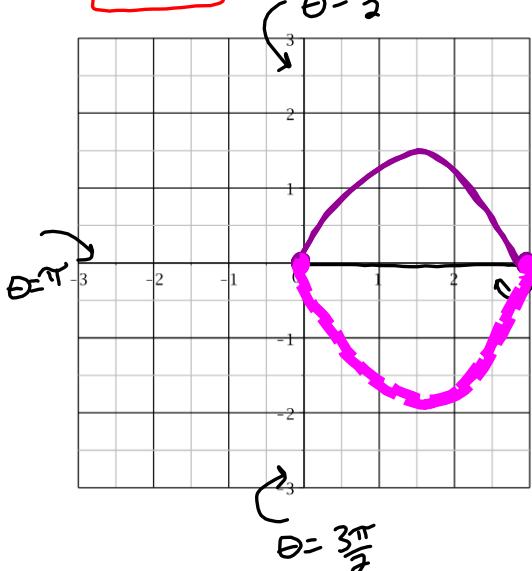


d.)  $0 \leq r \leq 2, \frac{\pi}{2} \leq \theta \leq \pi$



7. Sketch the graph of each polar equation.

a.)  $r = 3 \cos \theta$

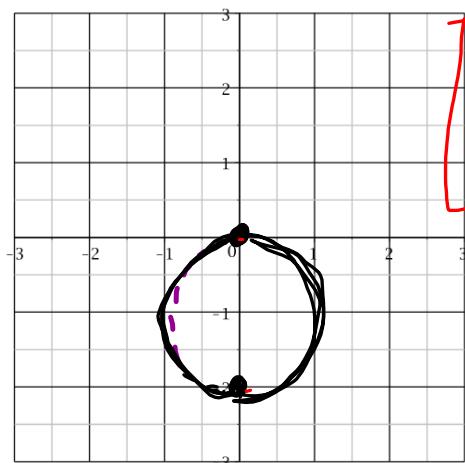


$\theta$	$r = 3 \cos \theta$	$(r, \theta)$
$0 \rightarrow \frac{\pi}{2}$	$3 \rightarrow 0$	$(3, 0) \rightarrow (0, \frac{\pi}{2})$
$\frac{\pi}{2} \rightarrow \pi$	$0 \rightarrow -3$	$(0, \frac{\pi}{2}) \rightarrow (-3, \pi)$

$r = a \cos \theta$  is a circle  
 $r = a \sin \theta$  is a circle

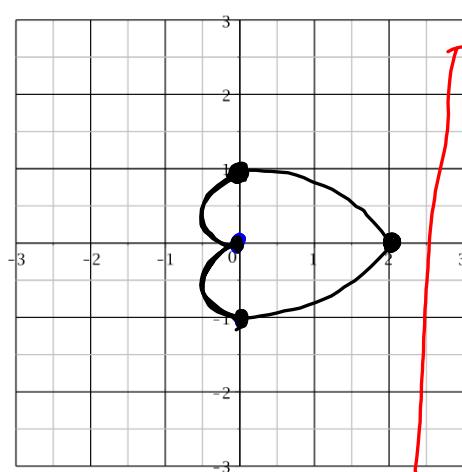
Full circle is generated  
for  $0 \leq \theta \leq \pi$

b.)  $r = -2 \sin \theta$      $\theta = \frac{\pi}{2}$



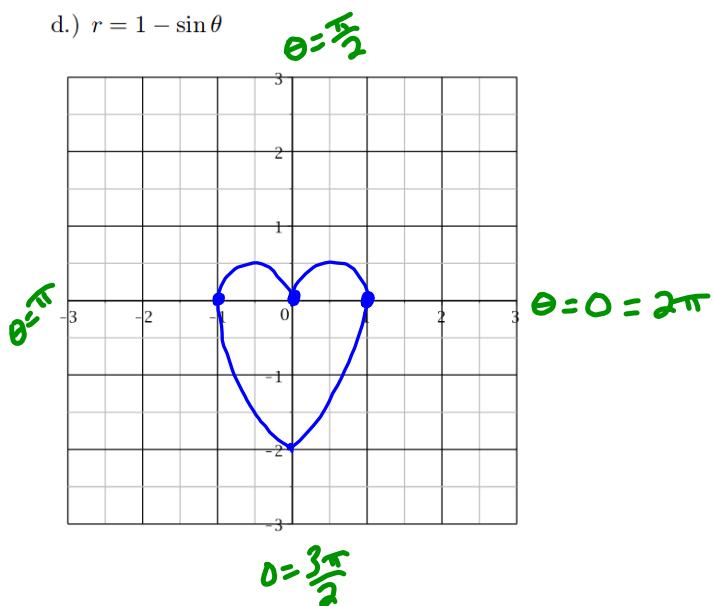
$\theta$	$r = -2 \sin \theta$	$(r, \theta)$
$0 \rightarrow \frac{\pi}{2}$	$0 \rightarrow -2$	$(0, 0) \rightarrow (-2, \frac{\pi}{2})$
$\frac{\pi}{2} \rightarrow \pi$	$-2 \rightarrow 0$	$(-2, \frac{\pi}{2}) \rightarrow (0, \pi)$

c.)  $r = 1 + \cos \theta$



$\theta$	$r = 1 + \cos \theta$	$(r, \theta)$
$0 \rightarrow \frac{\pi}{2}$	$2 \rightarrow 1$	$(2, 0) \rightarrow (1, \frac{\pi}{2})$
$\frac{\pi}{2} \rightarrow \pi$	$1 \rightarrow 0$	$(1, \frac{\pi}{2}) \rightarrow (0, \pi)$
$\pi \rightarrow \frac{3\pi}{2}$	$0 \rightarrow 1$	$(0, \pi) \rightarrow (1, \frac{3\pi}{2})$
$\frac{3\pi}{2} \rightarrow 2\pi$	$1 \rightarrow 2$	

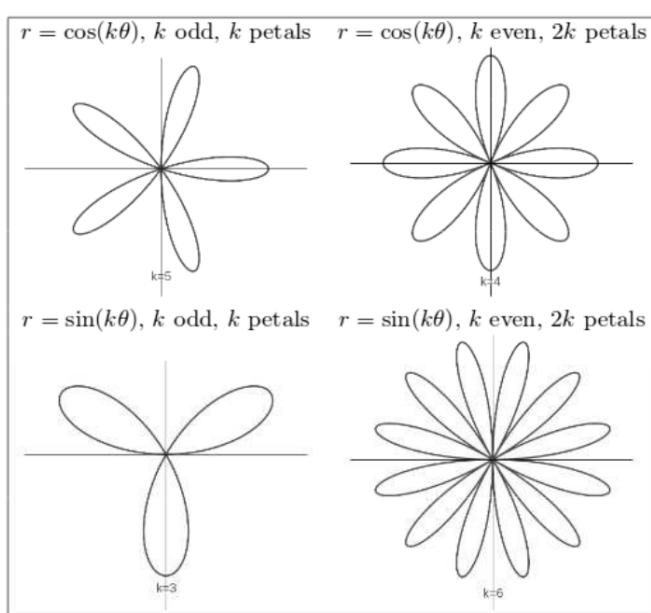
cardioid generated     $0 \leq \theta \leq 2\pi$



$$\begin{array}{c|c}
 \theta & r = 1 - \sin \theta \\
 \hline
 0 \rightarrow \frac{\pi}{2} & 1 \rightarrow 0 \\
 \frac{\pi}{2} \rightarrow \pi & 0 \rightarrow 1 \\
 \pi \rightarrow \frac{3\pi}{2} & 1 \rightarrow 2 \\
 \frac{3\pi}{2} \rightarrow 2\pi & 2 \rightarrow 1
 \end{array}$$

Petals! For  $r = \sin(kx)$  or  $r = \cos(kx)$ , if  $k$  is odd, the graph has  $k$  petals. If  $k$  is even, the graph has  $2k$  petals. Note: If  $k=1$ , there is one petal, hence a circle as we saw in 7 a.) and 7 b.).

Summary of Roses:



$$r = \sin(k\theta)$$

$$r = \cos(k\theta)$$

$$\sin(3\theta) \quad 3 \text{ petals}$$

$$\cos(8\theta) \quad 16 \text{ petals}$$

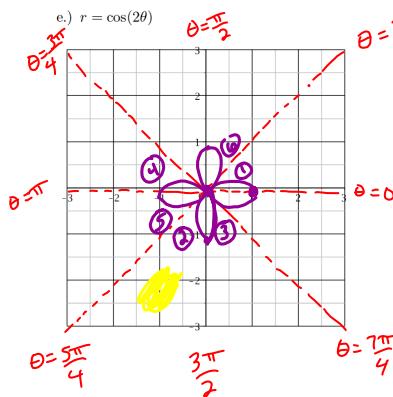
$$k=1 \rightarrow 1 \text{ petal}$$

$$r = \sin \theta \quad \text{circle}$$

$$r = \cos \theta \quad \text{circle}$$

For even  $k$ ,  $\theta$  must go from  $0 \rightarrow 2\pi$

For odd  $k$ ,  $\theta$  must go from  $0 \rightarrow \pi$



$$2\theta = 0 \quad \theta = 0$$

$$2\theta = \frac{\pi}{2} \quad \theta = \frac{\pi}{4}$$

$$2\theta = \pi \quad \theta = \frac{\pi}{2}$$

$$2\theta = \frac{3\pi}{2} \quad \theta = \frac{3\pi}{4}$$

$$2\theta = 2\pi \quad \theta = \pi$$

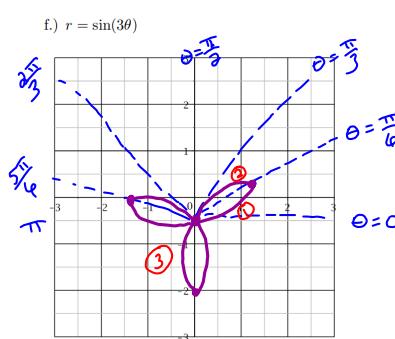
$$2\theta = 2\pi + \frac{\pi}{2} = \frac{5\pi}{2} \rightarrow \theta = \frac{5\pi}{4}$$

$$2\theta = 2\pi + \pi = 3\pi \rightarrow \theta = \frac{3\pi}{2}$$

$$2\theta = 2\pi + \frac{3\pi}{2} = \frac{7\pi}{2} \rightarrow \theta = \frac{7\pi}{4}$$

$$2\theta = 2\pi + 2\pi = 4\pi \rightarrow \theta = 2\pi$$

$\theta$	$r = \cos(2\theta)$
$0 \rightarrow \frac{\pi}{4}$	$1 \rightarrow 0$ ①
$\frac{\pi}{4} \rightarrow \frac{\pi}{2}$	$0 \rightarrow -1$ ②
$\frac{\pi}{2} \rightarrow \frac{3\pi}{4}$	$-1 \rightarrow 0$ ③
$\frac{3\pi}{4} \rightarrow \pi$	$0 \rightarrow 1$ ④
$\pi \rightarrow \frac{5\pi}{4}$	$1 \rightarrow 0$ ⑤
$\frac{5\pi}{4} \rightarrow \frac{3\pi}{2}$	$0 \rightarrow -1$ ⑥
$\frac{3\pi}{2} \rightarrow \frac{7\pi}{4}$	$-1 \rightarrow 0$ ⑦



θ odd → θ goes 0 → 2π

$$3\theta = 0 \rightarrow \theta = 0$$

$$3\theta = \frac{\pi}{2} \rightarrow \theta = \frac{\pi}{6}$$

$$3\theta = \pi \rightarrow \theta = \frac{\pi}{3}$$

$$3\theta = 2\pi \rightarrow \theta = \frac{2\pi}{3}$$

$$3\theta = 2\pi + \frac{\pi}{2} = \frac{5\pi}{2} \rightarrow \theta = \frac{5\pi}{6}$$

$$3\theta = 2\pi + \pi = 3\pi \rightarrow \theta = \pi$$

$\theta$	$r = \sin(3\theta)$
$0 \rightarrow \frac{\pi}{6}$	$0 \rightarrow 1$ ①
$\frac{\pi}{6} \rightarrow \frac{\pi}{3}$	$1 \rightarrow 0$ ②
$\frac{\pi}{3} \rightarrow \frac{\pi}{2}$	$0 \rightarrow -1$ ③
$\frac{\pi}{2} \rightarrow \frac{2\pi}{3}$	$-1 \rightarrow 0$
$\frac{2\pi}{3} \rightarrow \frac{5\pi}{6}$	$0 \rightarrow 1$
$\frac{5\pi}{6} \rightarrow \pi$	$1 \rightarrow 0$

