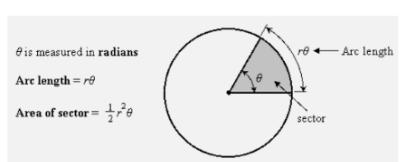


Section 10.4 Areas and Lengths in Polar Coordinates

In this section, we develop a formula for the area of a region whose boundary is given by a polar equation. Recall the area of a sector is $A = \frac{1}{2}r^2\theta$, where r is the radius and θ is the radian measure of the central angle.



$$A = \frac{1}{2} r^2 \theta$$

Let R be the region, illustrated the figure below, bounded by the polar curve $r = f(\theta)$ and the rays $\theta = a$ and $\theta = b$.

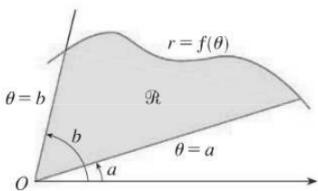


FIGURE 2

The area A of the polar region R is $A = \int_a^b \frac{1}{2} [f(\theta)]^2 d\theta = \int_a^b \frac{1}{2} r^2 d\theta$.

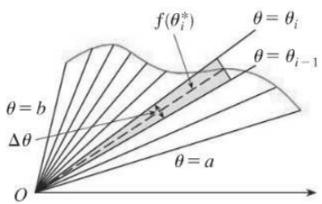


FIGURE 3

1. Find the area of the region that is bounded by $r = \frac{1}{\theta}$ that lies in the sector $\frac{\pi}{2} \leq \theta \leq 2\pi$.

$$\begin{aligned} A &= \int_{\frac{\pi}{2}}^{2\pi} \frac{1}{2} r^2 d\theta = \frac{1}{2} \int_{\frac{\pi}{2}}^{2\pi} \frac{1}{\theta^2} d\theta \\ &= \frac{1}{2} \left(-\frac{1}{\theta} \right) \Big|_{\frac{\pi}{2}}^{2\pi} \\ &= \frac{1}{2} \left(-\frac{1}{2\pi} + \frac{1}{\frac{\pi}{2}} \right) \end{aligned}$$

2. Find the area of the region that is bounded by $r = \sin \theta + \cos \theta$ that lies in the sector $0 \leq \theta \leq \pi$.

$$\begin{aligned}
 A &= \int_0^{\pi} \frac{1}{2} r^2 d\theta = \frac{1}{2} \int_0^{\pi} (\sin \theta + \cos \theta)^2 d\theta \\
 &= \frac{1}{2} \int_0^{\pi} (\sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta) d\theta \\
 &= \frac{1}{2} \int_0^{\pi} (1 + 2 \sin \theta \cos \theta) d\theta \\
 &\quad \begin{matrix} u = \sin \theta \\ du = \cos \theta d\theta \\ \int 2u du = u^2 = \sin^2 \theta \end{matrix} \\
 &= \frac{1}{2} \left(\theta + \sin^2 \theta \right) \Big|_0^{\pi} = \frac{1}{2} (\pi + 0) = \boxed{\frac{\pi}{2}}
 \end{aligned}$$

3. Sketch the curve $r = 2 \sin \theta$ and find the area it encloses.

$k=1 \rightarrow \text{odd} \rightarrow \text{whole circle generated for } 0 \leq \theta \leq \pi$

$$\begin{aligned}
 A &= \int_0^{\pi} \frac{1}{2} r^2 d\theta = \frac{1}{2} \int_0^{\pi} 4 \sin^2 \theta d\theta \\
 &= \frac{1}{2} \int_0^{\pi} 4 \cdot \frac{1}{2} (1 - \cos 2\theta) d\theta \\
 &= \theta - \frac{1}{2} \sin 2\theta \Big|_0^{\pi} \\
 &= \boxed{-\pi}
 \end{aligned}$$

4. Find the area of the shaded region.

$$\begin{aligned}
 A &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} r^2 d\theta = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4 + 3\sin\theta)^2 d\theta \\
 &= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (16 + 24\sin\theta + 9\sin^2\theta) d\theta \\
 &= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(16 + 24\sin\theta + \frac{9}{2}(1 - \cos 2\theta) \right) d\theta
 \end{aligned}$$

Correct Answer $\frac{41\pi}{4}$

5. ~~Graph~~ the curve $r = 1 - \sin\theta$ and find the area it encloses.

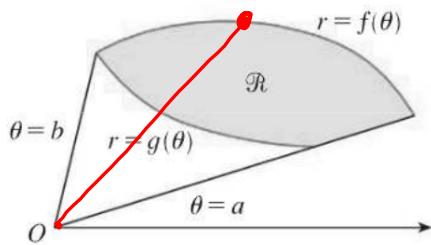
cardioid generated by $0 \leq \theta \leq 2\pi$

$$\begin{aligned}
 A &= \int_0^{2\pi} \frac{1}{2} r^2 d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} (1 - \sin\theta)^2 d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} (1 - 2\sin\theta + \sin^2\theta) d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} \left(1 - 2\sin\theta + \frac{1}{2}(1 - \cos 2\theta) \right) d\theta
 \end{aligned}$$

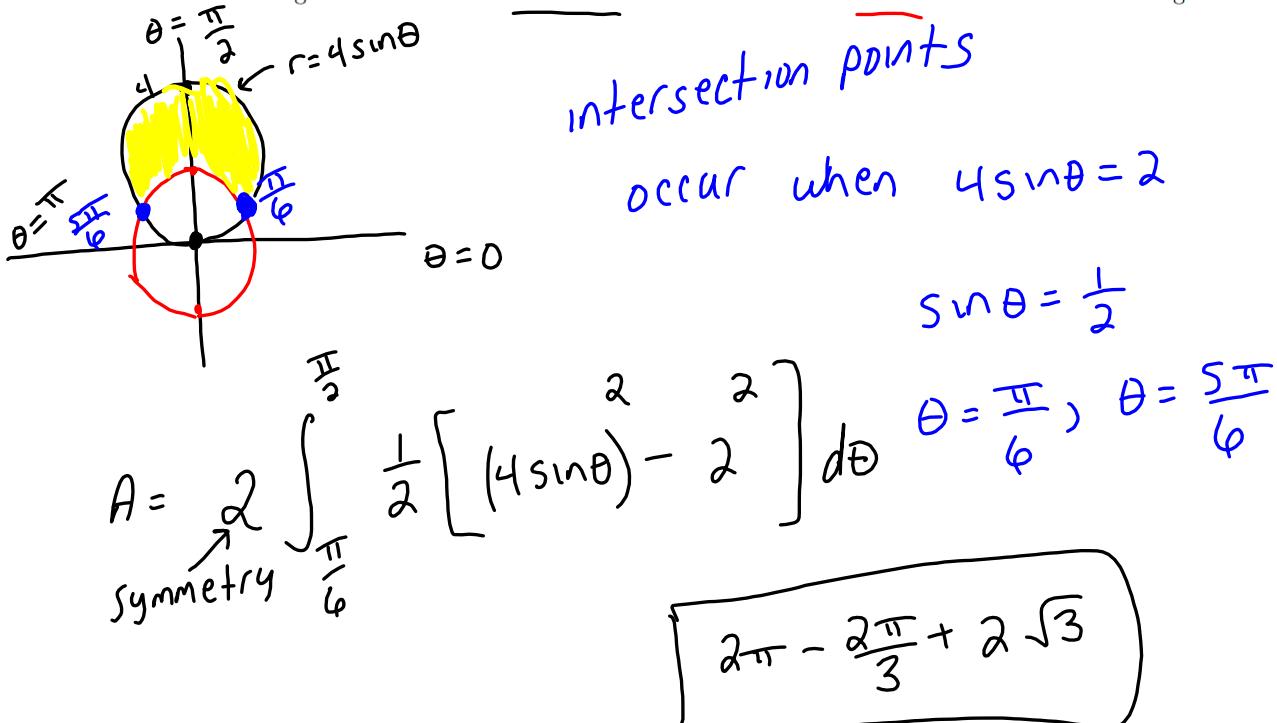
$$= \boxed{\frac{3\pi}{2}}$$

Let R be the region that is bounded by $f(\theta)$ and $g(\theta)$, $a \leq \theta \leq b$, as shown below. Then the area of R is

$$A = \int_a^b \frac{1}{2} ([f(\theta)]^2 - [g(\theta)]^2) d\theta$$

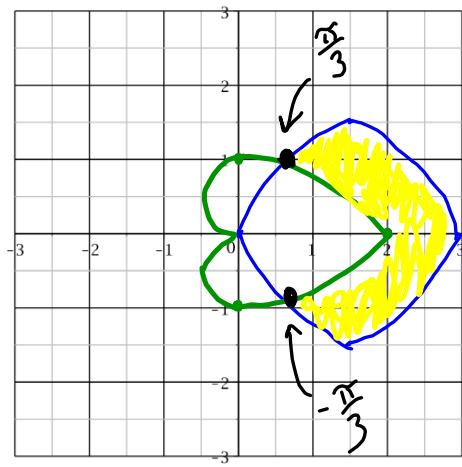


6. Sketch the region inside the circle $r = 4 \sin \theta$ and outside the circle $r = 2$. Find the area of the region.



7. Sketch the region inside the circle $r = 3 \cos \theta$ and outside the cardioid $r = 1 + \cos \theta$. Find the area of the region.

Do not evaluate the integral.



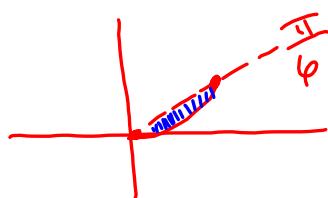
$$3 \cos \theta = 1 + \cos \theta$$

$$2 \cos \theta = 1$$

$$\theta = \frac{\pi}{3}, -\frac{\pi}{3}$$

$$A = 2 \int_0^{\frac{\pi}{3}} \frac{1}{2} \left(9 \cos^2 \theta - (1 + \cos \theta)^2 \right) d\theta$$

8. Set up but do not evaluate an integral that gives the area of one loop of the polar curve $r = \sin(3\theta)$.



$$3\theta = 0$$

$$3\theta = \frac{\pi}{2} \rightarrow \theta = \frac{\pi}{6}$$

By symmetry

$$A = 2 \int_0^{\frac{\pi}{6}} \frac{1}{2} (\sin 3\theta)^2 d\theta$$

will not be on final, but is in webassign

The length of the polar curve $r = f(\theta)$ $a \leq \theta \leq b$, is $L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$.

9. Find the length of the polar curve $r = \theta^2$, $0 \leq \theta \leq \frac{\pi}{2}$

$$\begin{aligned}
 L &= \int_0^{\frac{\pi}{2}} \sqrt{\theta^4 + 4\theta^2} d\theta \\
 &= \int_0^{\frac{\pi}{2}} \sqrt{\theta^2(\theta^2 + 4)} d\theta && u = \theta^2 + 4 \\
 &&& du = 2\theta d\theta \\
 &= \frac{1}{2} \frac{2}{3} (\theta^2 + 4)^{\frac{3}{2}} \Big|_0^{\frac{\pi}{2}} && \frac{3}{2} \\
 &= \frac{1}{3} \left(\left(\frac{\pi^2}{4} + 4 \right)^{\frac{3}{2}} - 4^{\frac{3}{2}} \right)
 \end{aligned}$$

10. Set up but do not evaluate an integral that gives the length of one loop of the curve $r = \cos(2\theta)$.

