Section 11.1 Sequences

Definition: A **sequence** is an **ordered** list of numbers a_1, a_2, a_3,...,a_n, ... The sequence $\{a_1, a_2, a_3, .., a_{100}, ..\}$ is often denoted by $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}$. The domain of a sequence is a subset of all non negative integers, usually indexed by n. We will deal with infinite sequences, and so each term, a_n, will have a successor, a_{n+1}.

1. Write out the first 3 terms of the sequence $\left\{\dfrac{n+1}{n+3}\right\}_{n=2}^{\infty}$.

2. Write out the first 4 terms of the recursive sequence:

   (a) $a_1 = 6$, $a_{n+1} = \dfrac{a_n}{n}$

   (b) $a_1 = 1$, $a_{n+1} = 2a_n - 3$

3. Find a general formula for the sequence, assuming the pattern continues. Assume n begins with 1.

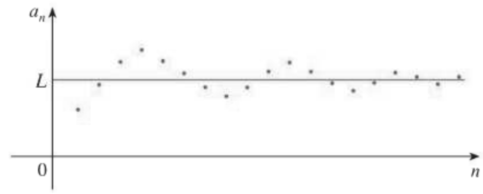
   (a) $\{\frac{1}{4}, -\frac{2}{9}, \frac{3}{16}, -\frac{4}{25}, ...\}$

   (b) $\{1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, ...\}$

   (c) $\{3, 2, \frac{4}{3}, \frac{8}{9}, \frac{16}{27}, ...\}$

   (d) $\{-3, 5, -7, 9, ...\}$

Definition: If $\lim_{n\to\infty} a_n = L$, then we say the sequence $\{a_n\}$ **converges** to L. If $\lim_{n\to\infty} a_n = \infty$ or does not exist, then we say the sequence $\{a_n\}$ **diverges**.



4. Determine whether the following sequences converge or diverge. If it converges, find the limit. If it diverges, explain why.

(a) $a_n = \sqrt{\dfrac{4n+3}{7n+6}}$

(b) $a_n = \arccos\left(\dfrac{-n+1}{2n+3}\right)$

(c) $a_n = \dfrac{1}{3}\ln(3n+1) - \dfrac{1}{3}\ln(4n)$

(d) $a_n = \dfrac{n}{(\ln n)^2}$

(e) $a_n = e^{\frac{n-n^2}{2n+1}}$

(f) $a_n = \dfrac{n}{n!}$

(g) $a_n = \{0, 2, 0, 2, 0, 2, ...\}$

Sequences whose terms alternate signs: For alternating sequences, the sequence converges if and only if the **absolute value** of the sequence goes to zero. Moreover, if If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$

Illustration of an alternating sequence that converges to 0: Note: **The absolute value of the terms of the sequence go to zero!**

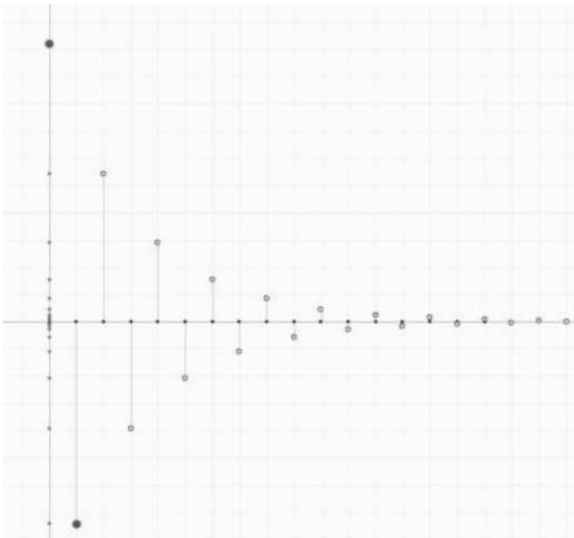
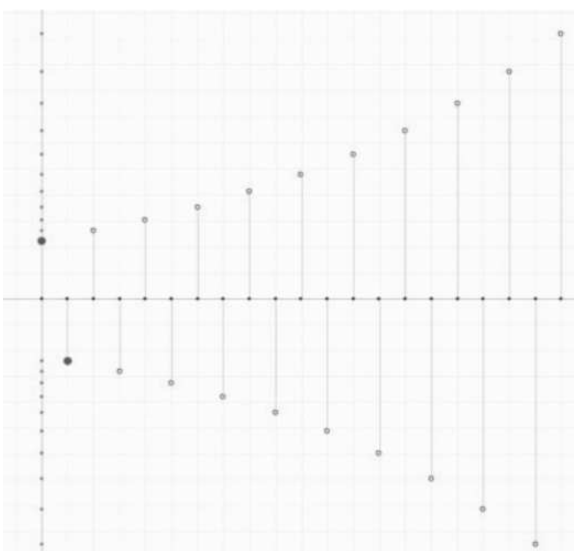


Illustration of an alternating sequence that diverges: **The absolute value of the terms of the sequence do not go to zero!**



5. Find the limit of the sequence, or if it diverges, explain why.

(a) $a_n = \dfrac{(-1)^n n}{2n+2}$

(b) $a_n = 3 + (-0.4)^n$

(c) $a_n = \cos(n)$ and $a_n = \cos\left(\dfrac{1}{n}\right)$

(d) $a_n = \dfrac{\sin(n)}{n}$

Definition: We say a sequence is bounded below if there is a number N so that $a_n \geq N$ for all n. We say a sequence is bounded above if there is a number M so that $a_n \leq M$ for all n. If a_n is bounded both above and below, then we say the sequence is bounded.

6. Determine whether the sequence is bounded:

   a.) $a_n = \left\{ \dfrac{1}{n^2} \right\}_{n=1}^{\infty}$

   b.) $a_n = \left\{ \dfrac{n^2}{n+1} \right\}_{n=1}^{\infty}$

   c.) $a_n = \{4^{-n}\}_{n=0}^{\infty}$

Definition: We say a sequence a_n is increasing if $a_n < a_{n+1}$ from some point on. We say a sequence a_n is decreasing if $a_n > a_{n+1}$ from some point on. If a sequence is either increasing or decreasing, then we say the sequence is monotonic.

7. Determine whether following sequences are increasing, decreasing, or not monotonic.

a.) $a_n = \dfrac{3}{n+5}$

b.) $a_n = e^{-n}$

c.) $a_n = 2 + \dfrac{1}{n}$

(d) $a_n = \cos\left(\dfrac{n\pi}{2}\right)$

8. Is it true that a decreasing bounded sequence must converge?

9. For the following **recursive sequences**:

   (a) $a_1 = 3$, $a_{n+1} = 2 + \dfrac{a_n}{3}$. Find the first 5 terms of the sequence. Find the limit of the sequence.

   (b) $a_1 = 2$, $a_{n+1} = 1 - \dfrac{1}{a_n}$. Find the first 5 terms of the sequence. Find the limit of the sequence.

   (c) $a_1 = 4$, $a_{n+1} = \dfrac{12}{8 - a_n}$ is bounded and decreasing. Find the next two terms of the sequence and find the limit.