Section 11.10 Taylor and Maclaurin Series

Let $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$

$$= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + c_5(x-a)^5 + ... + c_i(x-a)^i + ...$$

Substituting $x = a$ into $f(x)$ gives $\boxed{c_0 = f(a)}$

Take the derivative of $f(x)$:

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + 5c_5(x-a)^4 + ...$$

Substituting $x = a$ into $f'(x)$ gives $\boxed{c_1 = f'(a)}$

Likewise,

$$f''(x) = 2c_2 + 2 \cdot 3c_3(x-a) + 3 \cdot 4c_4(x-a)^2 + 4 \cdot 5c_5(x-a)^3 + ...$$

Substituting $x = a$ into $f''(x)$ gives $f''(a) = 2c_2$, yielding $\boxed{c_2 = \frac{f''(a)}{2}}$

$$f'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4(x-a) + 3 \cdot 4 \cdot 5c_5(x-a)^2 + ...$$

Substituting $x = a$ into $f'''(x)$ gives $f'''(a) = 2 \cdot 3c_3 = 3!c_3$, yielding $\boxed{c_3 = \frac{f'''(a)}{3!}}$

Continuing in this manner, we find $\boxed{c_i = \frac{f^i(a)}{i!}}$

Thus, we define the **Taylor Series** for $f(x)$ about $x = a$ to be

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + ...$$

where $f^{(n)}(a)$ is the nth derivative of $f(x)$ at $x = a$.

1. Find the Taylor Series for f centered at $a = 5$ if $f^{(n)}(5) = \dfrac{(-2)^n n!}{7^n(n+5)}$. What is the radius of convergence of this Taylor Series?

2. Find $f^{(31)}(5)$ if $f(x) = \sum_{n=0}^{\infty} \dfrac{2^{n+1}(x-5)^n}{(n+2)!}$, that is the 31^{st} derivative of f at $x = 5$.

3. Find the Taylor Series for $f(x) = e^{2x}$ centered at $a = -1$. What is the associated radius of convergence?

4. Find the Taylor Series for $f(x) = \dfrac{1}{x}$ centered at $a = 7$. What is the associated radius of convergence?

5. Find the Taylor Series for $f(x) = \ln x$ centered at 2. What is the associated radius of convergence?

Definition: The Maclaurin Series for the function $f(x)$ is defined to be the Taylor Series about $x = 0$. That is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

where $f^{(n)}(0)$ is the nth derivative of $f(x)$ at $x = 0$.

Note: We have been dealing with Maclaurin series in section 11.8 and 11.9 since a Maclaurin series is just a power series centered at $a = 0$. Thus, in particular, in section 11.9 we were asked to find a power series representation for $\dfrac{1}{1-3x}$ and we found $\dfrac{1}{1-3x} = \sum_{n=0}^{\infty} 3^n x^n$. Since this series is centered at zero, it is also considered a Maclaurin series.

6. Find the Maclaurin Series for $f(x) = e^x$. What is the associated radius of convergence?

7. Find the Maclaurin Series for $f(x) = \cos x$. What is the associated radius of convergence?

8. Find the Maclaurin Series for $f(x) = \sin x$. What is the associated radius of convergence?

9. Find the Maclaurin Series for $f(x) = e^{x^2}$.

10. Evaluate the indefinite integral as an infinite series $\int x \cos \frac{x}{2}\, dx$.

11. Evaluate the indefinite integral as an infinite series $\int \sin(4x^2)\, dx$.

12. Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n (\pi)^{2n+1}}{3^{2n+1}(2n+1)!}$.

13. Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^{3n}}{n!}$.