

Section 11.10 Taylor and Maclaurin Series

$$\text{Let } f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

coefficients

$$c_n = \frac{f^{(n)}(a)}{n!}$$

$$= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + c_5(x-a)^5 + \dots + c_i(x-a)^i + \dots$$

Substituting $x = a$ into $f(x)$ gives $c_0 = f(a) = \frac{f^0(a)}{0!}$

Take the derivative of $f(x)$:

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + 5c_5(x-a)^4 + \dots$$

Substituting $x = a$ into $f'(x)$ gives $c_1 = f'(a) = \frac{f'(a)}{1!}$

Likewise,

$$f''(x) = 2c_2 + 2 \cdot 3c_3(x-a) + 3 \cdot 4c_4(x-a)^2 + 4 \cdot 5c_5(x-a)^3 + \dots$$

Substituting $x = a$ into $f''(x)$ gives $f''(a) = 2c_2$, yielding $c_2 = \frac{f''(a)}{2} = \frac{f''(a)}{2!}$

$$f'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4(x-a) + 3 \cdot 4 \cdot 5c_5(x-a)^2 + \dots$$

Substituting $x = a$ into $f'''(x)$ gives $f'''(a) = 2 \cdot 3c_3 = 3!c_3$, yielding $c_3 = \frac{f'''(a)}{3!}$

Continuing in this manner, we find $c_i = \frac{f^i(a)}{i!}$

Thus, we define the **Taylor Series** for $f(x)$ about $x = a$ to be

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

where $f^{(n)}(a)$ is the n th derivative of $f(x)$ at $x = a$.

1. Find the Taylor Series for f centered at $a = 5$ if $f^{(n)}(5) = \frac{(-2)^n n!}{7^n(n+5)}$. What is the radius of convergence of this Taylor Series?

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(5)}{n!} (x-5)^n \\ &= \sum_{n=0}^{\infty} \frac{(-2)^n n!}{7^n(n+5)n!} (x-5)^n \end{aligned}$$

To find R , do ratio test.
Find $R = \frac{7}{2}$

2. Find $f^{(31)}(5)$ if $f(x) = \sum_{n=0}^{\infty} \frac{2^{n+1}(x-5)^n}{(n+2)!}$, that is the 31st derivative of f at $x = 5$.

This is a Taylor series centered at 5, with coefficients $c_n = \frac{2^{n+1}}{(n+2)!} = \frac{f^{(n)}(5)}{n!}$

$$n=31 \quad \frac{2^{32}}{33!} = \frac{f^{31}(5)}{31!}$$

$$f^{31}(5) = \frac{31! \cdot 2^{32}}{33!} = \frac{31! \cdot 2^{32}}{(33)(32)(31!)}$$

$$f^{31}(5) = \frac{2^{32}}{(33)(32)}$$

3. Find the Taylor Series for $f(x) = e^{2x}$ centered at $a = -1$. What is the associated radius of convergence?

$$e^{2x} = \sum_{n=0}^{\infty} \frac{f^n(-1)}{n!} (x+1)^n$$

find this

$$e^{2x} = \sum_{n=0}^{\infty} \frac{2^n e}{n!} (x+1)^n$$

$$R = \infty$$

$$f(x) = e^{2x}$$

$$f'(x) = 2e^{2x}$$

$$f''(x) = 2^2 e^{2x}$$

$$f'''(x) = 2^3 e^{2x}$$

$$f^{(4)}(x) = 2^4 e^{2x}$$

$$f^{(n)}(-1) = 2^n e$$

4. Find the Taylor Series for $f(x) = \frac{1}{x}$ centered at $a = 7$. What is the associated radius of convergence?

$$\frac{1}{x} = \sum_{n=0}^{\infty} \frac{f^n(7)}{n!} (x-7)^n$$

$$\frac{1}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{7^{n+1} n!} (x-7)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{7^{n+1}} (x-7)^n$$

via Ratio Test
 $R = 7$

$$f(x) = \frac{1}{x}$$

$$f'(x) = -\frac{1}{x^2} = -x^{-2}$$

$$f''(x) = \frac{2}{x^3} = 2x^{-3}$$

$$f'''(x) = -\frac{3 \cdot 2}{x^4}$$

$$f^{(4)}(x) = \frac{4 \cdot 3 \cdot 2}{x^5}$$

$$f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}$$

$$f^{(n)}(7) = \frac{(-1)^n n!}{7^{n+1}}$$

$\sum_{n=0}^{\infty} a_n = a_0 + \sum_{n=1}^{\infty} a_n$

5. Find the Taylor Series for $f(x) = \ln x$ centered at 2. What is the associated radius of convergence?

$$\ln x = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!} (x-2)^n$$

$$\ln x = \frac{f^0(2)}{0!} (x-2)^0 + \sum_{n=1}^{\infty} \frac{f^{(n)}(2)}{n!} (x-2)^n$$

$$\ln x = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)(n-1)!}{2^n n!} (x-2)^n$$

$$= \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n n} (x-2)^n \quad R=2$$

$f(x) = \ln x \rightarrow f(2) = \ln 2$
 $f'(x) = \frac{1}{x}$
 $f''(x) = -\frac{1}{x^2}$
 $f'''(x) = \frac{2}{x^3}$
 $f^4(x) = -\frac{3 \cdot 2}{x^4}$
 $f^5(x) = \frac{4 \cdot 3 \cdot 2}{x^5}$
 $f^{(n)}(x) = \frac{(-1)^{n+1} (n-1)!}{x^n}$
 only for $n \geq 1$
 $f^{(n)}(2) = \frac{(-1)^{n+1} (n-1)!}{2^n}$

Definition: The Maclaurin Series for the function $f(x)$ is defined to be the Taylor Series about $x = 0$. That is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

where $f^{(n)}(0)$ is the n th derivative of $f(x)$ at $x = 0$.

Note: We have been dealing with Maclaurin series in section 11.8 and 11.9 since a Maclaurin series is just a power series centered at $a = 0$. Thus, in particular, in section 11.9 we were asked to find a power series representation for $\frac{1}{1-3x}$ and we found $\frac{1}{1-3x} = \sum_{n=0}^{\infty} 3^n x^n$. Since this series is centered at zero, it is also considered a Maclaurin series.

6. Find the Maclaurin Series for $f(x) = e^x$. What is the associated radius of convergence?

center is 0

$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad R = \infty$$

$f(x) = e^x$
 $f'(x) = e^x$
 $f^{(n)}(x) = e^x$
 $f^{(0)}(0) = e^0 = 1$

7. Find the Maclaurin Series for $f(x) = \cos x$. What is the associated radius of convergence?

$$\cos x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$\begin{aligned} f &= \cos x \\ f' &= -\sin x \\ f'' &= -\cos x \\ f^4 &= \sin x \\ f^4 &= \cos x \end{aligned}$$

$$\cos x = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^4(0)x^4}{4!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \dots$$

$$\boxed{\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}}$$

$R = \infty$

8. Find the Maclaurin Series for $f(x) = \sin x$. What is the associated radius of convergence?

$$\boxed{\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad R = \infty}$$

9. Find the Maclaurin Series for $f(x) = e^{x^2}$.

$$\text{Know: } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

10. Evaluate the indefinite integral as an infinite series $\int x \cos \frac{x}{2} dx$.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \rightarrow x \cos \frac{x}{2} = x \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n}}{(2n)!}$$

$$\int x \cos \frac{x}{2} dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^n (2n)!} dx = x \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n (2n)!}$$

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2^n (2n)! (2n+2)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^n (2n)!}$$

11. Evaluate the indefinite integral as an infinite series $\int \sin(4x^2) dx$.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\int \sin(4x^2) dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n (4x^2)^{2n+1}}{(2n+1)!} dx$$

$$= \int \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n+1} x^{4n+2}}{(2n+1)!} dx$$

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n+1} x^{4n+3}}{(2n+1)! (4n+3)}$$

$$12. \text{ Find the sum of the series } \sum_{n=0}^{\infty} \frac{(-1)^n (\pi)^{2n+1}}{3^{2n+1} (2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n+1}}{(2n+1)!} = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$13. \text{ Find the sum of the series } \sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^{3n}}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n \left(\frac{x^3}{n!}\right)}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-2x^3)^n}{n!} = \boxed{0}$$

