Section 11.11 Taylor Polynomials

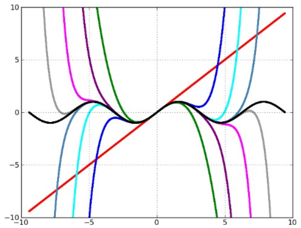
Definition: Let $f(x)$ be a function. Recall the Taylor Series for $f(x)$ at $x = a$ is

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$. A partial sum of a Taylor Series is called a Taylor Polynomial. More specifically, the n^{th} degree **Taylor Polynomial** for $f(x)$ at $x = a$ is

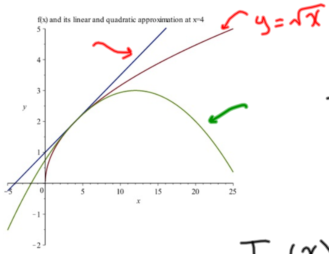
$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

where $f^{(i)}(a)$ is the ith derivative of $f(x)$ at $x = a$.

The Taylor polynomial at $x = a$ is useful in approximating a function near $x = a$. The first degree Taylor polynomial at $x = a$ is the same as the tangent line to $f(x)$ at $x = a$, and the second degree Taylor polynomial at $x = a$ is the same as the quadratic approximation to $f(x)$ at $x = a$. The higher the degree, the better approximation.



1. Find both the first and second degree Taylor polynomial for $f(x) = \sqrt{x}$ at $x = 4$.



center

$$T_1(x) = f(4) + \frac{f'(4)}{1!}(x-4)^1$$

$$T_2(x) = f(4) + \frac{f'(4)}{1!}(x-4) + \frac{f''(4)}{2!}(x-4)^2$$

$f(x) = \sqrt{x}$ $\quad f(4) = 2$

$f'(x) = \frac{1}{2\sqrt{x}}$ $\quad f'(4) = \frac{1}{4}$

$$\boxed{T_1(x) = 2 + \frac{1}{4}(x-4)}$$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

$$T_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$$

$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} = -\frac{1}{4x^{3/2}}$ $\quad f''(4) = -\frac{1}{4(4)^{3/2}} = -\frac{1}{32}$

2. Find the third degree Taylor Polynomial for $f(x) = xe^x$ at $x = 2$.

$$T_3(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3$$

$f(x) = xe^x$ $\boxed{f(2) = 2e^2}$

$f'(x) = e^x + xe^x$ $\boxed{f'(2) = e^2 + 2e^2 = 3e^2}$

$f''(x) = e^x + e^x + xe^x = 2e^x + xe^x$ $f''(2) = 2e^2 + 2e^2 = 4e^2$

$f'''(x) = 2e^x + e^x + xe^x = 3e^x + xe^x$ $\boxed{f'''(2) = 3e^2 + 2e^2 = 5e^2}$

$$T_3(x) = 2e^2 + 3e^2(x-2) + \frac{4e^2}{2}(x-2)^2 + \frac{5e^2}{6}(x-2)^3$$

3. Find $T_8(x)$ for $f(x) = x^2e^{-2x^3}$ at $x = 0$.

can use maclaurin!

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$x^2e^{-2x^3} = x^2 \sum_{n=0}^{\infty} \frac{(-2x^3)^n}{n!}$$

$$= x^2 \sum_{n=0}^{\infty} \frac{(-2)^n x^{3n}}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-2)^n x^{3n+2}}{n!}$$

$T_8(x)$

$$= \boxed{\underbrace{x^2}_{n=0} - \underbrace{2x^5}_{n=1} + \underbrace{\frac{4x^8}{2}}_{n=2}} + \cdots$$