

Section 11.1 Sequences

Definition: A **sequence** is an **ordered** list of numbers $a_1, a_2, a_3, \dots, a_n, \dots$. The sequence $\{a_1, a_2, a_3, \dots, a_{100}, \dots\}$ is often denoted by $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}$. The domain of a sequence is a subset of all non negative integers, usually indexed by n . We will deal with infinite sequences, and so each term, a_n , will have a successor, a_{n+1} .

1. Write out the first 3 terms of the sequence $\left\{ \frac{n+1}{n+3} \right\}_{n=2}^{\infty} = \left\{ \frac{3}{5}, \frac{4}{6}, \frac{5}{7}, \dots \right\}$

$$\begin{array}{c} \frac{3}{5}, \frac{4}{6}, \frac{5}{7}, \dots \\ \uparrow \quad \uparrow \quad \uparrow \\ a_2 \quad a_3 \quad a_4 \end{array}$$

2. Write out the first 4 terms of the recursive sequence:

(a) $a_1 = 6, a_{n+1} = \frac{a_n}{n}$

$$a_1 = 6$$

$$a_2 = \frac{a_1}{1} = \frac{6}{1} = 6$$

$$a_3 = \frac{a_2}{2} = \frac{6}{2} = 3$$

$$a_4 = \frac{a_3}{3} = \frac{3}{3} = 1$$

(b) $a_1 = 1, a_{n+1} = 2a_n - 3$

$$a_1 = 1$$

$$a_2 = 2a_1 - 3 = 2(1) - 3 = -1$$

$$a_3 = 2a_2 - 3 = 2(-1) - 3 = -5$$

$$a_4 = 2a_3 - 3 = 2(-5) - 3 = -13$$

3. Find a general formula for the sequence, assuming the pattern continues; assume n starts with 1.

$$(a) \left\{ \frac{1}{4}, -\frac{2}{9}, \frac{3}{16}, -\frac{4}{25}, \dots \right\} = \left\{ \frac{1}{2^2}, \frac{-2}{3^2}, \frac{3}{4^2}, \frac{-4}{5^2}, \dots \right\}_{n+1}$$

if terms alternate signs, (-1) or $\underline{(-1)}$

$$a_n = \frac{(-1)^{n+1}}{(n+1)^2}$$

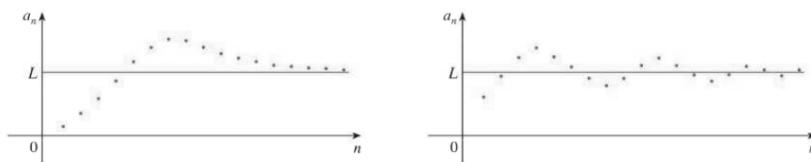
$$(b) \left\{ 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \dots \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3 \cdot 2}, \frac{1}{4 \cdot 3 \cdot 2}, \frac{1}{5 \cdot 4 \cdot 3 \cdot 2}, \dots \right\} = \frac{1}{n!}$$

$$(c) \left\{ 3, 2, \frac{4}{3}, \frac{8}{9}, \frac{16}{27}, \dots \right\} = \left\{ \overset{\swarrow}{3}, \overset{\swarrow}{2}, \overset{2}{\frac{2}{3}}, \overset{3}{\frac{2}{3^2}}, \overset{4}{\frac{2}{3^3}} \right\}$$

$$= \left\{ \frac{2^{n-1}}{3^{n-2}} \right\}_{n=1}^{\infty} = \frac{2^0}{3^{-1}}, \frac{2^1}{3^0}, \frac{2^2}{3^1}, \frac{2^3}{3^2}, \dots$$

$$(d) \left\{ -3, 5, -7, 9, \dots \right\} a_n = (-1)^n (2n+1)$$

Definition: If $\lim_{n \rightarrow \infty} a_n = L$, then we say the sequence $\{a_n\}$ **converges** to L . If $\lim_{n \rightarrow \infty} a_n = \infty$ or does not exist, then we say the sequence $\{a_n\}$ **diverges**.



4. Determine whether the following sequences converge or diverge. If it converges, find the limit. If it diverges, explain why.

$$(a) a_n = \sqrt{\frac{4n+3}{7n+6}} \quad \lim_{n \rightarrow \infty} \sqrt{\frac{4n+3}{7n+6}} = \sqrt{\lim_{n \rightarrow \infty} \frac{4n+3}{7n+6}} = \sqrt{\frac{4}{7}}$$

$$(b) a_n = \arccos\left(\frac{-n+1}{2n+3}\right) \quad \arccos\left(\lim_{n \rightarrow \infty} \frac{-n+1}{2n+3}\right) = \arccos\left(-\frac{1}{2}\right) \\ = \frac{2\pi}{3}$$

$$(c) a_n = \frac{1}{3} \ln(3n+1) - \frac{1}{3} \ln(4n)$$

$$\ln(\infty) = \infty \quad \lim_{n \rightarrow \infty} \left(\frac{1}{3} \ln(3n+1) - \frac{1}{3} \ln(4n) \right) = \infty - \infty \quad ?$$

$$\lim_{n \rightarrow \infty} \frac{1}{3} \left(\ln\left(\frac{3n+1}{4n}\right) \right) = \frac{1}{3} \ln\left(\frac{3}{4}\right)$$

$$(d) a_n = \frac{n}{(\ln n)^2}$$

$$\lim_{n \rightarrow \infty} \frac{n}{(\ln n)^2} \stackrel{\infty}{\cancel{\infty}} \stackrel{L}{=} \lim_{n \rightarrow \infty} \frac{1}{2(\ln n)\left(\frac{1}{n}\right)}$$

$$= \lim_{n \rightarrow \infty} \frac{n}{2\ln n} \stackrel{\infty}{\cancel{\infty}}$$

$$\stackrel{L}{=} \lim_{n \rightarrow \infty} \frac{1}{2 \cdot \frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n}{2} = \infty$$

diverges

will finish Thursday!

Section 11.1 continued

$$(e) a_n = e^{\frac{n-n^2}{2n+1}} \stackrel{\left(\frac{n-n^2}{2n+1}\right)}{\cancel{e}} = e^{\lim_{n \rightarrow \infty} \left(\frac{n-n^2}{2n+1}\right)}$$

$\lim_{n \rightarrow \infty} \frac{n-n^2}{2n+1} \stackrel{-\infty}{\cancel{-\infty}}$

$$\stackrel{L}{=} \lim_{n \rightarrow \infty} \frac{1-2n}{2} = -\infty$$

$$= e^{-\infty} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0$$

$$= \boxed{0}$$

$$(f) a_n = \frac{n}{n!}$$

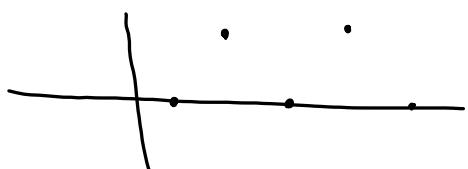
$$\lim_{n \rightarrow \infty} \frac{n}{n!} = \lim_{n \rightarrow \infty} \frac{x}{x(n-1)!} = \boxed{0}$$

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$= 5 \cdot 4!$$

$$= 5 \cdot 4 \cdot 3!$$

$$(g) a_n = \{0, 2, 0, 2, 0, 2, \dots\}$$

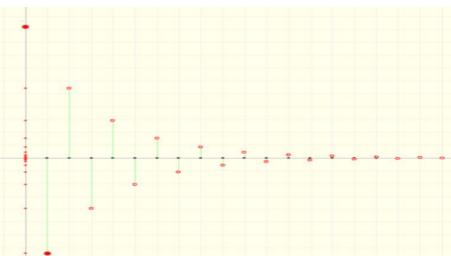


$$\lim_{n \rightarrow \infty} a_n$$

does not exist
by oscillation

diverges

Sequences whose terms alternate signs: For alternating sequences, the sequence converges if and only if the absolute value of the sequence goes to zero. Moreover, if $\lim_{n \rightarrow \infty} |a_n| = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$. Illustration of an alternating sequence that converges to 0: Note: The absolute value of the terms of the sequence go to zero!

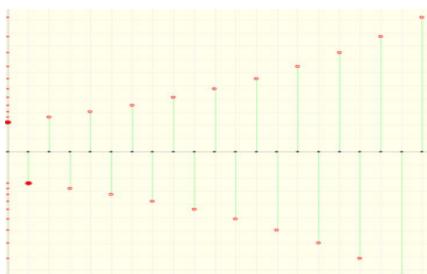


Def: An alternating sequence is a sequence whose terms alternate signs.

$$\leftarrow \lim_{n \rightarrow \infty} |a_n| = 0$$

sequence a_n converges to 0.

Illustration of an alternating sequence that diverges: The absolute value of the terms of the sequence do not go to zero!



$$\leftarrow \lim_{n \rightarrow \infty} |a_n| \neq 0$$

sequence a_n diverges by oscillation.

5. Find the limit of the sequence, or if it diverges, explain why.

$$(a) a_n = \frac{(-1)^n n}{2n+2}$$

since a_n is an alternating sequence,

$$\text{evaluate } \lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^n n}{2n+2} \right|$$

so $\frac{(-1)^n n}{2n+2}$ diverges by oscillation.

$$= \lim_{n \rightarrow \infty} \frac{n}{2n+2}$$

$$(b) a_n = 3 + (-0.4)^n$$

$$\lim_{n \rightarrow \infty} (3 + (-0.4)^n)$$

$$\cancel{\lim_{n \rightarrow \infty} 3} + \cancel{\lim_{n \rightarrow \infty} (-0.4)^n}$$

$$= 3 + 0$$

= 3 converges to 3

$$= \frac{1}{2} \neq 0! \quad \begin{matrix} \lim_{n \rightarrow \infty} a^n = \infty \text{ if } |a| > 1 \\ \lim_{n \rightarrow \infty} a^n = 0 \text{ if } |a| < 1 \end{matrix}$$

alternating, so look at

$$\lim_{n \rightarrow \infty} |(-0.4)^n| = \lim_{n \rightarrow \infty} (0.4)^n$$

$$= \lim_{n \rightarrow \infty} \left(\frac{2}{5} \right)^n$$

$$= 0 \text{ since } \frac{2}{5} < 1$$

$\lim_{n \rightarrow \infty} \cos(n)$ dne by oscillation

$$\lim_{n \rightarrow \infty} \cos\left(\frac{1}{n}\right) = \cos\left(\lim_{n \rightarrow \infty} \frac{1}{n}\right) = \cos(0) = 1$$

converges to 1

$$(d) a_n = \frac{\sin(n)}{n}$$

$$\lim_{n \rightarrow \infty} \frac{\sin(n)}{n} = 0 \quad \text{since}$$

$\sin(n)$ is at most 1

$$\lim_{n \rightarrow \infty} \left| \frac{\sin(n)}{n} \right| \leq \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

Definition: We say a sequence is bounded below if there is a number N so that $a_n \geq N$ for all n . We say a sequence is bounded above if there is a number M so that $a_n \leq M$ for all n . If a_n is bounded both above and below, then we say the sequence is bounded.

6. Determine whether the sequence is bounded:

$$\text{a.) } a_n = \left\{ \frac{1}{n^2} \right\}_{n=1}^{\infty} = \left\{ 1, \frac{1}{4}, \frac{1}{9}, \dots \right\}$$

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} = 0 \quad 0 < a_n \leq 1$$

bounded below by 0
bounded above by 1 → sequence is bounded

$$\text{b.) } a_n = \left\{ \frac{n^2}{n+1} \right\}_{n=1}^{\infty} \quad \lim_{n \rightarrow \infty} \frac{n^2}{n+1} = \infty \rightarrow \text{not bounded}$$

$\left\{ \frac{1}{2}, \frac{4}{3}, \dots \right\}$ but is bounded below by $\frac{1}{2}$ but not bounded above, hence not bounded.

ex: $\left\{ (-4)^n \right\}$ $\begin{cases} \infty & n \text{ even} \\ -\infty & n \text{ odd} \end{cases}$

$$\text{c.) } a_n = \{4^{-n}\}_{n=0}^{\infty}$$

$$a_n = \left\{ \frac{1}{4^n} \right\}_{n=0}^{\infty} = \left\{ 1, \frac{1}{16}, \frac{1}{64}, \dots \right\}$$

bounded

$$0 < \frac{1}{4^n} \leq 1$$

Definition: We say a sequence a_n is increasing if $a_n < a_{n+1}$ from some point on. We say a sequence a_n is decreasing if $a_n > a_{n+1}$ from some point on. If a sequence is either increasing or decreasing, then we say the sequence is monotonic.

7. Determine whether following sequences are increasing, decreasing, or not monotonic.

a.) $a_n = \frac{3}{n+5}$

decreasing since $\frac{3}{n+5}$ gets smaller as n increases.

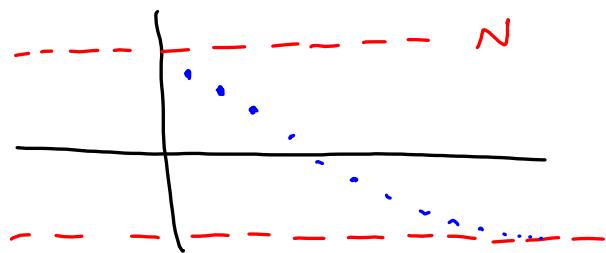
b.) $a_n = e^{-n} = \frac{1}{e^n}$ decreasing

c.) $a_n = 2 + \frac{1}{n}$ decreasing since $\frac{1}{n}$ decreases as $n \rightarrow \infty$

(d) $a_n = \cos\left(\frac{n\pi}{2}\right) = \left\{ \begin{array}{cccc} 0, & -1, & 0, & 1, \\ n=1 & n=2 & n=3 & n=4 \end{array}, \dots \right\}$

neither increasing nor decreasing \rightarrow non monotonic

8. Is it true that a decreasing bounded sequence must converge?



$$M < a_n < N$$

yes

9. For the following recursive sequences:

(a) $a_1 = 3$, $a_{n+1} = 2 + \frac{a_n}{3}$. Find the first 5 terms of the sequence. Find the limit of the sequence.

$$\begin{aligned} a_1 &= 3 \\ a_2 &= 2 + \frac{a_1}{3} = 2 + \frac{3}{3} = 3 \\ a_3 &= 2 + \frac{a_2}{3} = 2 + \frac{3}{3} = 3 \\ a_4 &= 2 + \frac{a_3}{3} = 2 + \frac{3}{3} = 3 \\ a_5 &= 2 + \frac{a_4}{3} = 2 + \frac{3}{3} = 3 \end{aligned}$$

$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} 3 = 3$
converges to 3

(b) $a_1 = 2$, $a_{n+1} = 1 - \frac{1}{a_n}$. Find the first 5 terms of the sequence. Find the limit of the sequence.

$$\begin{aligned} a_1 &= 2 \\ a_2 &= 1 - \frac{1}{a_1} = 1 - \frac{1}{2} = \frac{1}{2} \\ a_3 &= 1 - \frac{1}{a_2} = 1 - \frac{1}{\frac{1}{2}} = 1 - 2 = -1 \\ a_4 &= 1 - \frac{1}{a_3} = 1 - \frac{1}{-1} = 2 \\ a_5 &= a_2 = \frac{1}{2} \end{aligned}$$

$\lim_{n \rightarrow \infty} a_n$ due by oscillation
so it diverges.

(c) $a_1 = 4$, $a_{n+1} = \frac{12}{8-a_n}$ is bounded and decreasing. Find the next two terms of the sequence and find the limit.

$$\begin{aligned} a_1 &= 4 \\ a_2 &= \frac{12}{8-a_1} = \frac{12}{8-4} = 3 \\ a_3 &= \frac{12}{8-a_2} = \frac{12}{8-3} = \frac{12}{5} \end{aligned}$$

Find the limit of the sequence.

Suppose $\lim_{n \rightarrow \infty} a_n = L$. Find L .

$$a_{n+1} = \frac{12}{8-a_n}$$

$$\lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} \frac{12}{8-a_n}$$

$$\underbrace{\lim_{n \rightarrow \infty} a_{n+1}}_L = \frac{12}{8 - \lim_{n \rightarrow \infty} a_n} \quad \begin{aligned} \lim_{n \rightarrow \infty} a_n &= L \\ \lim_{n \rightarrow \infty} a_{n+1} &= L \end{aligned}$$

$$\begin{aligned} L &= \frac{12}{8-L} \\ L(8-L) &= 12 \\ 8L - L^2 &= 12 \\ 0 &= L^2 - 8L + 12 \\ 0 &= (L-6)(L-2) \\ L &= 6, L = 2 \end{aligned}$$

since first term was 4,
and sequence is decreasing,

$$L = 2$$