Before class starts, determine whether the following sequences converge or diverge. If it converges, find the limit. If it diverges, explain why.

1. $a_n = \frac{n}{4n+1}$

converges to $\frac{1}{4}$

2. $a_n = \frac{(-1)^n n}{4n+1}$ $\frac{1}{4}$

diverges by oscillation.

3. $a_n = \frac{(-1)^n n}{4n^2+1}$ $\rightarrow 0$

Section 11.2 Series

Definition: If we try to add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, ...\}$ together, we will get an expression of the form $a_1 + a_2 + a_3 + ... + a_n + ... + ...$ which is called a **series** and is denoted, for short, by the symbol $\sum_{n=1}^{\infty} a_n$, or Σa_n. Does it even make sense to talk about the sum of infinitely many terms?

It would be impossible to find a finite sum for the series $\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + 5 + ... + n...$ because if we start adding terms, we get cumulative sums (called **partial sums**) s_1, s_2, s_3,..,s_n,...that grow without bound.

$s_1 = a_1 = 1$

$s_2 = a_1 + a_2 = 1 + 2 = 3$

$s_3 = a_1 + a_2 + a_3 = 1 + 2 + 3 = 6$

$s_4 = a_1 + a_2 + a_3 + a_4 = 1 + 2 + 3 + 4 = 10$

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$S_n = \{a_1, a_1 + a_2, a_1 + a_2 + a_3, ...\}$

a_n = terms being added

S_n = partial sum

$s_n = a_1 + a_2 + a_3 + a_4 + ... + a_n = 1 + 2 + 3 + 4 + ... + n = \frac{n(n+1)}{2}$ (This can be proved by induction)

Note that $\lim_{n\to\infty} s_n = \lim_{n\to\infty}(1 + 2 + 3 + ... + n) = \lim_{n\to\infty} \frac{n(n+1)}{2} = \infty$, thus $\sum_{n=1}^{\infty} n = \infty$.

Now let's look at the series $\boxed{\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + ...}$

and we look at the **sequence of partial sums**:

$s_1 = \frac{1}{2} = 0.5$

$\lim_{n\to\infty} \frac{1}{2^n} = 0$

$s_2 = \frac{1}{2} + \frac{1}{2^2} = 0.75$

$s_3 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} = 0.875$

$s_4 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} = 0.9375$

$s_5 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} = 0.96875$

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$s_{100} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + ... + \frac{1}{2^{100}} = \frac{1267650600228229401496703205375}{1267650600228229401496703205376}$:) Looks like 1!

This sequence appears to be approaching 1, suggesting the **limit** of the **sequence of partial sums** is **converging to** 1, and we write $\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{n\to\infty} s_n = 1$

Now we will formally define the terminology used on the previous page.

The definition of convergence or divergence of a series: If $\sum_{n=1}^{\infty} a_n = S$, where S is finite, then we say the series **converges** and its **sum** is S. If $\sum_{n=1}^{\infty} a_n = \infty$ or does not exist, then we say the series **diverges**.

How can we find the sum of a series? We do this by first finding a formula for the sequence of partial sums.

Definition: **The sequence of partial sums** is the sequence whose terms are the cumulative sums of the series.

Consider $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ...a_n + ...$

We will construct the **sequence of partial sums**, $\{s_n\} = \{s_1, s_2, s_3, ..., ...\}$, as follows:

$s_1 = a_1$ **Called the first partial sum**

$S_n = \{a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots a_1 + a_2 + \cdots + a_n, \}$

$s_2 = a_1 + a_2$ **Called the second partial sum**

$s_3 = a_1 + a_2 + a_3$ **Called the third partial sum**

Therefore a general formula for s_n, the n^{th} term of the sequence, is

$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + ... + a_n$ **Called the *nth* partial sum**

Note: For series that do not begin at an index of one, for example $\sum_{n=2}^{\infty} a_n$, we call $s_1 = a_2$ the first partial sum, $s_2 = a_2 + a_3$ the second partial sum, etc. **Thus in general, the *nth* partial sum is the sum of the first n terms, regardless of where the series begins.**

This is the definition of the sum of a series! **LEARN IT!!**

$$\boxed{\boxed{\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n} = \lim_{n\to\infty} \sum_{i=1}^{n} a_i}$$

$S_n =$ "sum of first n terms" $= a_1 + a_2 + \cdots + a_n$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (a_1 + a_2 + \cdots + a_n) = \sum_{n=1}^{\infty} a_n$$

1. Find the first 5 terms in the sequence of partial sums the series $\sum_{n=1}^{\infty}(1)$. Does the series coverge?

~~common mistake: $\sum_{n=1}^{\infty}(1) = 1$~~

$a_n = 1$

$S_1 = a_1 = 1$

$S_2 = a_1 + a_2 = 1 + 1 = 2$

$S_3 = a_1 + a_2 + a_3 = 1 + 1 + 1 = 3$

$S_4 = 4$

$S_5 = 5$

$\{S_n\} = \{1, 2, 3, 4, 5, \ldots\}$

series diverges because $\lim_{n\to\infty} S_n = \infty$

2. Find the first 5 terms in the sequence of partial sums the series $\sum_{n=1}^{\infty}(-1)^n$. Does the series coverge?

$a_n = (-1)^n$

$S_1 = a_1 = (-1)^1 = -1$

$S_2 = a_1 + a_2 = (-1)^1 + (-1)^2 = -1 + 1 = 0$

$S_3 = a_1 + a_2 + a_3 = (-1)^1 + (-1)^2 + (-1)^3 = -1 + 1 - 1 = -1$

$S_4 = 0$ $\quad \{S_n\} = \{-1, 0, -1, 0, \ldots\}$

$S_5 = -1$

$\lim_{n\to\infty} S_n = \text{dne} \rightarrow$ oscillates between -1 & 0

$\sum_{n=1}^{\infty}(-1)^n$ dne

Test for Divergence: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Intuitively, this makes sense because if $\lim_{n\to\infty} a_n \neq 0$, then we are continuing to add terms together that are not getting smaller, hence $\sum_{n=1}^{\infty} a_n$ must diverge.

NOTE: The converse is not necessarily true!: If $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=1}^{\infty} a_n$ **does not necessarily converge**. Therefore if you find that $\lim_{n\to\infty} a_n = 0$, then **the test for divergence fails** and thus another test must be applied. The classic example of a series that does not converge is the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. However, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ DOES converge. We will show this in the next section.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \infty$$

$$\lim_{n\to\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \text{finite}$$

3. What can we conclude about the following series, if anything, using the test for divergence?

a.) $\sum_{n=1}^{\infty} \frac{n}{5n+9}$

$a_n = \frac{n}{5n+9}$

T.D. If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=?}^{\infty} a_n$ diverges

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{5n+9} = \frac{1}{5} \neq 0$, $\boxed{\sum_{n=1}^{\infty} \frac{n}{5n+9} \text{ diverges}}$

b.) $\sum_{n=2}^{\infty} \cos n$

$a_n = \cos n$

$\lim_{n\to\infty} \cos n$ dne $\neq 0$ $\boxed{\sum_{n=2}^{\infty} \cos n \text{ diverges}}$

c.) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

$a_n = \frac{n}{n^2+1}$

$\lim_{n\to\infty} \frac{n}{n^2+1} = 0 \rightarrow$ T.D. fails and is therefore inconclusive.

d.) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

$a_n = \frac{\ln n}{n}$

$\lim_{n\to\infty} \frac{\ln n}{n} \quad \frac{\infty}{\infty} \stackrel{L}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0 \rightarrow$ T.D. inconclusive

4. If $\sum_{n=1}^{\infty} a_n$ converges, what, if anything, can be said about $\lim_{n\to\infty} a_n$?

$\lim_{n\to\infty} a_n = 0$

Because if $\lim_{n\to\infty} a_n \neq 0$ $\sum_{n=1}^{\infty} a_n$ would diverge by T.D.

Finding the sum of a series. Recall if $\{s_n\}$ is the sequence of partial sume of the series $\sum_{n=1}^{\infty} a_n$, then $\boxed{\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n}$.

$a_n =$ individual terms being added

$S_n =$ sum of first n terms

In other words, the **sum of a series** is the **limit of the sequence of partial sums.**

5. If the n^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n$ is $s_n = \frac{n+1}{2n+4}$, find:

given

a.) The sum of the first 5 terms.

$S_n =$ "sum of first n terms" $= \frac{n+1}{2n+4}$

$S_5 =$ sum of first 5 terms $= \frac{6}{14}$

b.) Does the series converge or diverge? If it converges, what is the sum?

$\hookrightarrow \sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n+1}{2n+4} = \frac{1}{2}$

converges, sum $= \frac{1}{2}$

c.) What is a_1? What is a_{10}?

Recall: $S_n = a_1 + a_2 + \cdots + a_n$

$S_1 = a_1$ Given $S_n = \frac{n+1}{2n+4}$ so $S_1 = \frac{2}{6}$

so $\boxed{a_1 = \frac{2}{6} = \frac{1}{3}}$

$S_{10} = \cancel{a_1} + \cancel{a_2} + \cdots + \cancel{a_9} + a_{10}$

$- \quad S_9 = \cancel{a_1} + \cancel{a_2} + \cdots + \cancel{a_9}$

$S_{10} - S_9 = a_{10}$

$\boxed{\frac{11}{24} - \frac{10}{22} = a_{10}}$

d.) Find a general formula for a_n.

$S_n - S_{n-1} = \boxed{\frac{n+1}{2n+4} - \frac{n-1+1}{2(n-1)+4} = a_n}$

6. If the n^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n$ is $s_n = e^{1/n}$, does the series converge? Support your answer.

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} e^{\frac{1}{n}} = e^{0} = 1$$

different question: what about $\sum_{n=1}^{\infty} e^{\frac{1}{n}}$? diverges by TD because $\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\frac{1}{n}} = 1 \neq 0$

Definition: A **telescoping series** is a series of the form $\sum_{n=1}^{\infty} (a_{n+i} - a_n)$ for some integer $i \geq 1$. Telescoping series can be identified by expanding the sum to see if an infinate number of terms cancel, and if they do, what is the end behavior?

7. Determine whether the following series converges or diverges. If it converges, find the sum. If it diverges, explain why.

a.) $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim_{n\to\infty} S_n$

$S_n = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$

$$= \underbrace{1 - \cancel{\tfrac{1}{2}}}_{a_1} + \underbrace{\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}}_{a_2} + \underbrace{\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}}}_{a_3} + \cancel{\tfrac{1}{4}} - \cancel{\tfrac{1}{5}} + \cancel{\ }\cdots \cancel{\ } + \underbrace{\cancel{\tfrac{1}{n-1}} - \cancel{\tfrac{1}{n}}}_{a_{n-1}} + \underbrace{\cancel{\tfrac{1}{n}} - \tfrac{1}{n+1}}_{a_n}$$

$$\boxed{S_n = 1 - \frac{1}{n+1}}$$

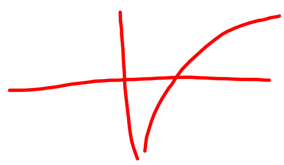
$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = \boxed{1}$$

sum of series

b.) $\sum_{n=1}^{\infty} \ln \frac{n+1}{n+2} = \sum_{n=1}^{\infty} \left[\ln(n+1) - \ln(n+2)\right]$

$S_n = \ln 2 - \cancel{\ln 3} + \cancel{\ln 3} - \cancel{\ln 4} + \cdots + \cancel{\ln n} - \cancel{\ln(n+1)} + \cancel{\ln(n+1)} - \ln(n+2)$

$S_n = \ln 2 - \ln(n+2)$



$S_n = \ln \frac{2}{n+2}$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln \frac{2}{n+2} = \ln\left(\lim_{n\to\infty} \cancel{\frac{2}{n+2}}^{\,0}\right) = \ln(\text{"}0\text{"}) = -\infty$$

Series diverges!

Section 11.2 (continued)

c.) $\displaystyle\sum_{\boxed{n=3}}^{\infty}\left(\cos\frac{1}{n+3}-\cos\frac{1}{n+4}\right)$

$S_n = a_3 + a_4 + \dots + a_n$

$$S_n = \underline{\cos\tfrac{1}{6}} - \cancel{\cos\tfrac{1}{7}} + \cancel{\cos\tfrac{1}{7}} - \cancel{\cos\tfrac{1}{8}} + \cancel{\dots} + \cancel{\cos\tfrac{1}{n+3}} - \underline{\cos\tfrac{1}{n+4}}$$

$$\boxed{S_n = \cos\frac{1}{6} - \cos\frac{1}{n+4}}$$

$$\sum_{n=3}^{\infty}\left(\cos\frac{1}{n+3} - \cos\frac{1}{n+4}\right) = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\cos\frac{1}{6} - \cancel{\cos\frac{1}{n+4}}\right)$$

$\cos 0 = 1$

$$= \boxed{\cos\frac{1}{6} - 1} \leftarrow \text{sum!}$$

d.) $\displaystyle\sum_{\underline{n=1}}^{\infty}\frac{7}{n(n+2)}$

$\underline{PFD}$. $\displaystyle\frac{7}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2} = \frac{\frac{7}{2}}{n} - \frac{\frac{7}{2}}{n+2}$

$7 = A(n+2) + Bn$

$n=0 \longrightarrow 7 = A(2) \longrightarrow A = \frac{7}{2}$

$n=-2 \longrightarrow 7 = B(-2) \longrightarrow B = -\frac{7}{2}$

$$\sum_{n=1}^{\infty}\left(\frac{\frac{7}{2}}{n} - \frac{\frac{7}{2}}{n+2}\right) = \frac{7}{2}\boxed{\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right)}$$

$\uparrow$ Find S_n

$S_n = a_1 + a_2 + a_3 + \dots + a_{n-1} + a_n$

$$= 1 - \cancel{\tfrac{1}{3}} + \tfrac{1}{2} - \cancel{\tfrac{1}{4}} + \cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{5}} + \cancel{\tfrac{1}{4}} - \cancel{\tfrac{1}{6}} + \dots + \underbrace{\cancel{\tfrac{1}{n-1}} - \tfrac{1}{n+1}}_{a_{n-1}} + \underbrace{\cancel{\tfrac{1}{n}} - \tfrac{1}{n+2}}_{a_n}$$

$$\boxed{S_n = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}} \longrightarrow \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right) = \lim_{n\to\infty}\left(\frac{3}{2} - \cancelto{0}{\frac{1}{n+1}} - \cancelto{0}{\frac{1}{n+2}}\right)$$

$$= \boxed{\frac{3}{2}}$$

$$\text{so } \sum_{n=1}^{\infty}\frac{7}{n(n+2)} = \frac{7}{2}\left(\frac{3}{2}\right)$$

$$= \boxed{\frac{21}{4}}$$

Definition: A **geometric series** is a series of the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + ... + ar^{n-1} + ar^n + ...$$

The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ will converge if $|r| < 1$ and will diverge if $|r| \geq 1$. Moreover, if $|r| < 1$, then $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$.

Note that not all series begin at 1, and not all powers of r are $n-1$. No matter the situation, a is always the first term of the sum of the series.

Note that not all series begin at 1, and not all powers of r are $n-1$. No matter the situation, a is always the first term of the sum of the series.

Proof: Let's form the sequence of partial sums for the series $\sum_{n=1}^{\infty} ar^{n-1}$.

$$s_n = \sum_{i=1}^{n} ar^{i-1} = a + ar + ar^2 + ar^3 + ... + ar^{n-1}$$

$$rs_n = \sum_{i=1}^{n} ar^{i} = ar + ar^2 + ar^3 + ar^4 + ... + ar^{n}$$

$$s_n - rs_n = a - ar^n$$

$$s_n(1-r) = a(1-r^n), \text{ thus } s_n = \frac{a(1-r^n)}{1-r}.$$

8. Determine whether the following geometric series converge or diverge. If it converges, find the sum. If it diverges, explain why.

a.) $\sum_{n=1}^{\infty} 5\left(\frac{2}{7}\right)^n$ $\quad r = \frac{2}{7} \quad |r| < 1$ will converge

$$\sum_{n=1}^{\infty} 5\left(\frac{2}{7}\right)\left(\frac{2}{7}\right)^{n-1} = \sum_{n=1}^{\infty} \frac{10}{7}\left(\frac{2}{7}\right)^{n-1} = \frac{a}{1-r}$$

$a = \frac{10}{7}, \quad r = \frac{2}{7}$

$$= \frac{\frac{10}{7}}{1-\frac{2}{7}} = \boxed{2}$$

sum $= \dfrac{\text{First Term}}{1-r}$

$$= \frac{\frac{10}{7}}{1-\frac{2}{7}}$$

b.) $\sum_{n=0}^{\infty} \frac{-4}{e^n} = \sum_{n=0}^{\infty} -4\left(\frac{1}{e^n}\right) = \sum_{n=0}^{\infty} -4\left(\frac{1}{e}\right)^n$ $\quad r = \frac{1}{e} \quad |r| < 1$

$$= \frac{a}{1-r} = \frac{-4}{1-\frac{1}{e}} \cdot \frac{e}{e} = \boxed{\frac{-4e}{e-1}}$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

c.) $\sum_{n=0}^{\infty} \frac{5^n}{(-3)^n} = \sum_{n=0}^{\infty} \left(\frac{-5}{3}\right)^n$ $\quad r = \frac{-5}{3}$ $\quad |r|>1$

$\boxed{\text{will diverge}}$

d.) $\sum_{n=1}^{\infty} \frac{(-1)^n + 3^{n-1}}{4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n} + \sum_{n=1}^{\infty} \frac{3^{n-1}}{4^n}$

$= \frac{-\frac{1}{4}}{1+\frac{1}{4}} + \frac{\frac{1}{4}}{1-\frac{3}{4}}$

short cut

$\frac{\text{first term}}{1-r}$

$$= \sum_{n=1}^{\infty} \left(-\frac{1}{4}\right)^n + \sum_{n=1}^{\infty} \frac{3^{n-1}}{4 \cdot 4^{n-1}}$$

$$= \sum_{n=1}^{\infty} \left(-\frac{1}{4}\right)^n + \sum_{n=1}^{\infty} \frac{1}{4}\left(\frac{3}{4}\right)^{n-1}$$

$|r| = \frac{3}{4} < 1$

$|r| = \frac{1}{4} < 1$

$$= \sum_{n=1}^{\infty} \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)^{n-1} + \sum_{n=1}^{\infty} \frac{1}{4}\left(\frac{3}{4}\right)^{n-1}$$

$$= \frac{-\frac{1}{4}}{1+\frac{1}{4}} + \frac{\frac{1}{4}}{1-\frac{3}{4}} = -\frac{1}{4}\left(\frac{4}{5}\right) + \frac{1}{4}(4)$$

$$= -\frac{1}{5} + 1 = \boxed{\frac{4}{5}}$$

e.) $7 + 2 + \frac{4}{7} + \frac{8}{49} + \ldots$ $\quad 7 + \frac{2}{7^0} + \frac{2^2}{7} + \frac{2^3}{7^2} + \cdots$ $\quad \frac{1}{7^{-1}}$

$$\sum_{n=0}^{\infty} \frac{2^n}{7^{n-1}} = \frac{7}{1-\frac{2}{7}}$$

$r = \frac{2}{7}$

$$= \frac{7}{\frac{5}{7}} = 7\left(\frac{7}{5}\right) = \frac{49}{5}$$

f.) $\sum_{n=3}^{\infty} \frac{5}{2^{2n}} = \sum_{n=3}^{\infty} \frac{5}{4^n}$

$r = \frac{1}{4}$

$$= \sum_{n=3}^{\infty} 5\left(\frac{1}{4}\right)^n = \sum_{n=3}^{\infty} 5\left(\frac{1}{4}\right)^3\left(\frac{1}{4}\right)^{n-3}$$

$\frac{\text{First term}}{1-r} = \frac{\frac{5}{64}}{1-\frac{1}{4}} = \frac{\frac{5}{64}}{1-\frac{1}{4}}$

9. Consider $\sum_{n=1}^{\infty} (x-5)^n$. Find the value(s) of x for which the series converges. Find the sum of the series for those values of x.